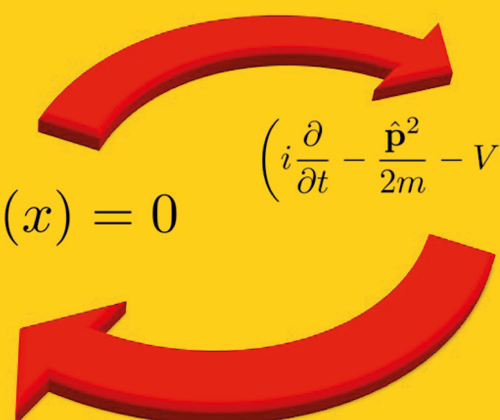
Materials 249

Antonio Pineda

Introduction to Quantum Field Theory



$$\left(i\frac{\partial}{\partial t}-\frac{\hat{\mathbf{p}}^2}{2m}-V(\hat{\mathbf{x}})\right)\psi(x)=0$$

$$\left(i\not{D}-m\right)\hat{\psi}(x)=0$$

UAB Universitat Autònoma de Barcelona
Servei de Publicacions

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Universitat Autònoma de Barcelona
Servei de Publicacions
Bellaterra, 2026

Primera edició: gener de 2026

Edició i impressió:
Servei de Publicacions
Universitat Autònoma de Barcelona
Edifici A. 08193 Bellaterra (Cerdanyola del Vallès). Spain
Tel. 93 581 10 22
sp@uab.cat
https://publicacions.uab.cat

Imprès a Espanya. Printed in Spain

ISBN (paper) 979-13-88031-08-3
ISBN (digital) 979-13-88031-09-0
Dipòsit legal: B 1838-2026

INTRODUCTION
TO
QUANTUM FIELD THEORY

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Preface

This book is largely based on the lecture notes from an introductory course on QFTs that I taught for several years during the final semester of the undergraduate Physics program at the *Universitat Autònoma de Barcelona*. In this regard, I am especially grateful to Érika Dengra, who made an initial transcription of these notes into LaTeX.

Chapter 1 presents some preliminary material. The first section introduces a small set of acronyms used throughout the book. The second section provides a brief summary comparing relativistic and NR notation. The third and fourth sections contain material that, depending on how the undergraduate physics program is structured, could be skipped or significantly condensed. This is material that students are expected to know beforehand. Otherwise, it represents the minimal background required to follow the course. The third section serves as a refresher on CFT, while the fourth revisits special relativity, with an emphasis on its group-theoretical aspects relevant to the construction of relativistic QFTs.

Chapter 2 focuses on QFTs for NR particles. Several aspects of the construction of a QFT and of its physical interpretation can already be explored within the NR framework. In particular, we will introduce the Fock space and the creation and annihilation operators associated with particles. We will also address common misconceptions regarding the distinction between fields (as operators) and the probabilistic interpretation of quantum states, and how this connects with standard quantum mechanics courses. This chapter will serve as a smooth transition toward the study of relativistic QFTs in later chapters.

Chapter 3 emphasizes that the building blocks of the Fock space in relativistic QFTs are the one-particle unitary irreducible representations of the theory's symmetry group—in our case, the Poincaré group.

Chapter 4 focuses on the construction of the KG field, both real and complex, and its connection to scalar particles. Noether's theorem is also introduced in this chapter.

Chapter 5 presents the general structure of the S-matrix for relativistic QFTs. Chapter 6 applies this framework to interacting scalar theories. These examples serve as templates for introducing Feynman diagrams and the corresponding Feynman rules.

Chapter 7 provides a field representation of the photon in position space.

With all the background material covered in the previous chapters, Chapter 8 presents a general set of Feynman rules applicable to most Lagrangians, and, certainly, to the physical processes discussed in this book.

Chapters 9, 10, and 11 delve into interacting theories involving photons and matter—whether scalar relativistic particles or NR ones. Finally, the bibliography lists the references that had the most significant influence on the development of this book.

Sections marked with an asterisk (*) indicate content that may be skipped or summarized in a more basic version of these lecture notes. This applies, for example, to the Wigner method, where one may simply use the results presented in Sec. 3.4. However, in a more advanced QFT course, such topics should be explored more thoroughly.

Based on my experience, the material presented in this book is more than sufficient for a one-semester undergraduate physics course. Therefore, we do not cover relativistic spin-$\frac{1}{2}$ particles (typically, but not exclusively, Dirac fields), nor do we address the regularization and renormalization of loop diagrams, nor gravity. However, we do include a final chapter on quantum gravity to briefly highlight its conceptual similarity with the photon case, and to show that we already possess the tools to compute Feynman diagrams involving gravitons coupled to matter and/or photons. A proper treatment of these topics—covered in the associated Master's course at the *Universitat Autònoma de Barcelona*—is postponed to the writing of the second part of this book.

The approach taken in this book differs in some respects from traditional QFT textbooks. I have always felt uneasy with treatments of QFT that did not provide a smooth connection to previous courses in NR quantum mechanics or CFTs, even though these should naturally emerge as limits of relativistic QFTs. These lecture notes make a deliberate effort to build that bridge, particularly with NR quantum mechanics. We also feel that, at times, introductory QFT courses aim too high, too far, too soon—trying to see the whole of the moon. Our goal here is more modest, and we aim to truly remain introductory, paying close attention to the details.

Antonio Pineda

UAB, January 2026

My way ...

1 Background Material

1.1 Table of Acronyms

For ease of reference, we list below the acronyms used in this book.

QFT: Quantum Field Theory
CFT: Classical Field Theory
CM: Classical Mechanics
NR: Non-relativistic
KG: Klein-Gordon
E-L: Euler-Lagrange
EoM: Equations of Motion
QED: Quantum Electrodynamics
NRQED: Non-relativistic Quantum Electrodynamics

1.2 Relativistic/Non-Relativistic Notation

The conventions we follow are similar to those of [7]. We take the $(1, -1, -1, -1)$ sign signature convention for the Minkowski metric. We will generically refer to $D = 1 + d$ for space-time dimensions and to d $(= 3)$ for space dimensions. By default, d-dimensional vectors in the NR notation will have the index up. We will use boldface to emphasize NR conventions, particularly when the notation is potentially ambiguous.

- $\partial_i = \frac{\partial}{\partial x^i} \equiv \boldsymbol{\nabla}^i$
- Four-vector notation: $A^\mu = (A^0, \vec{A}) \equiv (A^0, \mathbf{A})$
- Magnetic field

  $$\mathbf{B}^k = -\frac{1}{2}\epsilon^{kij}F_{ij} = -\frac{1}{2}\epsilon^{kij}(\partial^i A^j - \partial^j A^i) = \frac{1}{2}\epsilon^{kij}(\partial_i A^j - \partial_j A^i) = (\boldsymbol{\nabla} \times \mathbf{A})^k \,.$$

  We stress that the i sum index does not have a (-1) associated with it, as it is the relativistic convention for summation of one upper and one lower index. In the NR notation, we want all indices to be up and use Euclidean summation convention. This is why we introduce $\boldsymbol{\nabla}^i$.

- Electric field

  $$\mathbf{E}^i = -\partial^0 A^i + \partial^i A^0 = F^{i0} = -\partial_0 \mathbf{A}^i - \boldsymbol{\nabla}^i A_0 .$$

- Covariant derivative (minimal coupling with the electromagnetic field)

  $$iD_0 = i\partial_0 - eA_0, \qquad i\mathbf{D} = i\boldsymbol{\nabla} + e\mathbf{A} \quad \text{or} \quad iD_\mu = i\partial_\mu - eA_\mu \,, \tag{1.1}$$

  where the minus sign multiplying e is convention.

- Canonical coordinates for a four-vector. We take the coordinates to be A^μ. This means that we have to perform functional derivatives with respect tos $\dot{A}^\mu$.

  $$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = \frac{\partial \mathcal{L}}{\partial A_{\mu,0}} \,,$$

  and the Hamiltonian would read (if they were independent canonical coordinates)

  $$H = \int d^d\mathbf{x} \pi_\mu \dot{A}^\mu - L \,. \tag{1.2}$$

  If one only has spatial-vector components, one can go to a NR setup with the assignment: $\boldsymbol{\pi}^i \equiv \pi_i$. Note that a naive analysis may lead to a wrong sign for the spatial components.

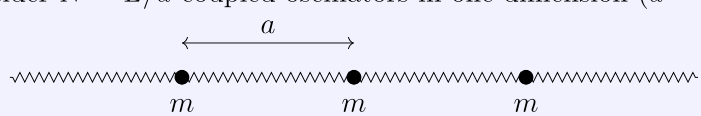
1.3 Classical Field Theory*

1.3.1 Motivation

We use the following example (see [1], for instance) to illustrate how CFT emerges in the continuum/scaling limit ($a \to 0$)

Example:

We consider $N = L/a$ coupled oscillators in one dimension ($d = 1$):



CM analysis

The kinetic term of each oscillator reads (we change notation: $x_i \to \psi_i$)

$$T_i = \frac{1}{2} m \dot{\psi}_i^2 \,, \tag{1.3}$$

where ψ_i is the relative position of the particle with respect to the minimum energy position. The sum of all kinetic energies reads

$$T = \frac{1}{2} \sum_{i=-N/2}^{N/2} m \dot{\psi}_i^2 \quad \xrightarrow{L \to \infty} \quad T = \frac{1}{2} \sum_{i \in \mathbb{Z}} m \dot{\psi}_i^2 \quad , \tag{1.4}$$

where in the last equation we have taken the infinite volume limit.

The total **potential energy** reads (in the infinite volume limit)

$$V = \frac{1}{2} \sum_{i \in \mathbb{Z}} k \left(\psi_{i+1} - \psi_i \right)^2 \quad . \tag{1.5}$$

As it is a conservative system, the **Lagrangian** reads

$$L = T - V \quad . \tag{1.6}$$

Using the E-L EoM:

$$\frac{\partial L}{\partial \psi_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}_i} = 0 \,, \tag{1.7}$$

one obtains

$$m \ddot{\psi}_i = k \left(\psi_{i+1} - 2\psi_i + \psi_{i-1} \right) = -k(\psi_i - \psi_{i+1}) - k(\psi_i - \psi_{i-1}) \quad . \tag{1.8}$$

Notation

From now on we switch to the following notation:

$$\psi_i \equiv \psi(t, ia) \equiv \psi(t, \mathbf{x}) \quad \text{or, sometimes,} \quad \psi(x). \tag{1.9}$$

This is a more suitable notation for field theory.

If we define $\mu \equiv \frac{m}{a}$ and $y \equiv ka$, our EoM reads

$$\Rightarrow \mu \frac{\partial^2}{\partial t^2}\psi(t,\mathbf{x}) = \frac{y}{a^2}\left[\psi(t,\mathbf{x}+a) - 2\psi(t,\mathbf{x}) + \psi(t,\mathbf{x}-a)\right]. \tag{1.10}$$

Now, recall that

$$f'(x) = \lim_{a\to 0} \frac{f(x+a) - f(x)}{a} \quad , \tag{1.11}$$

$$f''(x) = \lim_{a\to 0} \frac{f'(x+a) - f'(x)}{a} = \lim_{a\to 0} \frac{f(x+a) + f(x-a) - 2f(x)}{a^2} \quad . \tag{1.12}$$

At the continuum/scaling limit, where $a \to 0$ *and* $\begin{cases} \mu \to \text{finite} \\ y \to \text{finite} \end{cases}$, we obtain the wave equation ($\boldsymbol{\nabla}^2\psi = \left(\frac{\partial}{\partial x}\psi\right)^2$ in $d = 1$)

$$\mu \frac{\partial^2}{\partial t^2}\psi(t,\mathbf{x}) = y\boldsymbol{\nabla}^2\psi(t,\mathbf{x}) \ . \tag{1.13}$$

If we redefine $\mathbf{x}' \equiv \sqrt{\frac{\mu}{y}}\mathbf{x}$ and $\tilde{\psi} \equiv \sqrt{\mu}\psi$, we obtain

$$L = \frac{1}{2}\int d^d\mathbf{x}' \left\{ \left(\frac{\partial\tilde{\psi}}{\partial t}\right)^2 - \left(\boldsymbol{\nabla}\tilde{\psi}\right)^2 \right\} \sqrt{\frac{y}{\mu}}, \tag{1.14}$$

which, up to an overall constant, is the same Lagrangian that describes electromagnetic waves.

Overall, when taking the continuum/scaling limit, one obtains wave equations and fields emerge naturally.[1] The path we have followed is the following:

$$\text{(A)} \qquad \text{CM} \to \text{L} \to \text{EoM} \xrightarrow[a\to 0]{} \text{EoM (fields)}.$$

[1] Note that the continuum limit can also be taken for a system in a finite volume.

There is an alternative method, though. It consists of taking the continuum limit at the beginning:

$$\text{(B)} \qquad \text{CM} \underset{a\to 0}{\longrightarrow} \text{L (fields)} \xrightarrow[\text{derive it?}]{\text{How to}} \text{EoM (fields)}.$$

To follow this alternative path, we first need to write the Lagrangian in terms of fields using the following limit:

$$\sum_{i\in\mathbb{Z}} a \to \int d^d\mathbf{x}\,. \tag{1.15}$$

Then,

$$T = \frac{1}{2}\sum_{i\in\mathbb{Z}} a\mu\left(\partial_t\psi(t,ia)\right)^2 \underset{a\to 0}{\longrightarrow} T = \frac{1}{2}\mu\int d^d\mathbf{x}(\partial_t\psi(t,\mathbf{x}))^2, \tag{1.16}$$

$$V = \frac{1}{2}\sum_{i\in\mathbb{Z}} aY\left(\frac{\psi(t,ia+a)-\psi(t,ia)}{a}\right)^2 \underset{a\to 0}{\longrightarrow} V = \frac{1}{2}Y\int d^d\mathbf{x}\left(\boldsymbol{\nabla}\psi(t,\mathbf{x})\right)^2, \tag{1.17}$$

$$L = \int d^d\mathbf{x}\left\{\frac{\mu}{2}\dot{\psi}^2 - \frac{Y}{2}\left(\boldsymbol{\nabla}\psi\right)^2\right\} = L\left([\psi],[\dot{\psi}];t\right). \tag{1.18}$$

We can easily generalize the above discussion to $d = 3$ dimensions with the replacement $(\boldsymbol{\nabla}\psi)^2 \equiv (\partial_{x^1}\psi)^2 + (\partial_{x^2}\psi)^2 + (\partial_{x^3}\psi)^2$.

Overall, we have obtained an expression for the Lagrangian that depends on the values of the fields all over space —this is an example of a functional. Now, the question is how to work with functionals in order to derive the EoM. This topic is discussed in the following sections.

1.3.2 CM vs CFT (general discussion)

In CM, we work with generalized coordinates q_i, $i = 1,\ldots,n$, and the solutions are $q_i(t)$. To find these solutions, we write down a Lagrangian and apply the variational principle to the action to derive the EoM:

$$\ddot{q}_i(t) = f_i(\{q_j\},\{\dot{q}_s\};t) \quad \forall i\,. \tag{1.19}$$

One then solves these equations and gets $q_i(t)$ $\forall i$.

Analogy

In CFT: $i \to \mathbf{x}$ and $q \to \phi$. So, we have the following notation (more covariant)

$$x_i(t) \to \phi^A_{\mathbf{x}}(t) \equiv \phi^A(t,\mathbf{x})\,, \tag{1.20}$$

where A labels internal degrees of freedom or spin, if there are any. Consequently, the analogue of Eq. (1.19) would read

$$\ddot{\phi}_{\mathbf{x}} = F_{\mathbf{x}}\left([\phi],[\dot{\phi}];t\right), \tag{1.21}$$

where $[\phi] \to \phi_{\mathbf{y}}\ \forall\ \mathbf{y}$ and $[\dot{\phi}] \to \dot{\phi}_{\mathbf{z}}\ \forall\ \mathbf{z}$, indicates that the equation may depend on the values of the fields and their time derivatives throughout space. However, in the previous example, we had a local equation: $\partial_t^2\psi(t,x) = \frac{y}{\mu}\boldsymbol{\nabla}^2\psi$. Indeed, this is what will happen in most cases. Regardless, we will derive Eq. (1.21) using a variational principle as well.

We now explore the analogy at the level of the action and the E-L EoM.

CM: We consider an action

$$S = \int_{t_1}^{t_2} dt L(q,\dot{q},t) \longrightarrow \delta S = 0 \longrightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{1.22}$$

Recall that

$$\frac{d}{dt} \equiv D = \sum_{j=1}^{n}\left\{\dot{q}_j\frac{\partial}{\partial q_j} + \ddot{q}_j\frac{\partial}{\partial \dot{q}_j}\right\} + \frac{\partial}{\partial t}. \tag{1.23}$$

CFT:

$$S = \int_{t_1}^{t_2} dt L([\phi],[\dot{\phi}];t) \longrightarrow \delta S = 0 \xrightarrow[\text{analogy}]{?} \frac{\delta L}{\delta \phi_{\mathbf{x}}} - D\left(\frac{\delta L}{\delta \dot{\phi}_{\mathbf{x}}}\right) = 0, \tag{1.24}$$

where now

$$D = \int d^d\mathbf{y}\left(\dot{\phi}_{\mathbf{y}}\frac{\delta}{\delta \phi_{\mathbf{y}}} + \ddot{\phi}_{\mathbf{y}}\frac{\delta}{\delta \dot{\phi}_{\mathbf{y}}}\right) + \frac{\partial}{\partial t}. \tag{1.25}$$

Usually, we can write the Lagrangian in terms of a Lagrangian density[2]

$$L([\phi],[\dot{\phi}];t) = \int d^d\mathbf{x}\mathcal{L}\left(\phi(t,\mathbf{x}),\partial_i\phi(t,\mathbf{x}),\partial_{ij}\phi(t,\mathbf{x})\ldots,\dot{\phi}(t,\mathbf{x}),\partial_i\dot{\phi},\ldots;t,\mathbf{x}\right). \tag{1.27}$$

Observations:

[2]Note that

$$\partial_i\dot{\phi}(x^0,\mathbf{x}) \equiv \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^0}\phi(x^0,\mathbf{x}). \tag{1.26}$$

- Note that we can include an arbitrary number of spatial derivatives, even if we only have a single time derivative.
- Second-order time derivatives are not allowed if we intend to transition to the Hamiltonian formalism and quantize the theory.[3]
- Note that, if we impose Poincaré symmetry, the number of time and spatial derivatives in the Lagrangian are linked.

To quantify the above analogies we need some basic working knowledge of functionals. We provide this in the following section.

1.3.3 Functionals

A functional is a mapping

$$F[\ ] : \langle\psi\rangle \longrightarrow \begin{cases} \langle\psi\rangle \\ \text{or} \\ A \to \text{field}(\mathbb{R}, \mathbb{C}) \end{cases} \tag{1.28}$$

where $\langle\psi\rangle$ is a set of functions.

We have already seen examples of functionals. For example,

$$L([\phi], [\dot{\phi}]; t) \quad \text{or} \quad S[\phi]. \tag{1.29}$$

The functional derivative follows from the following definition (for arbitrary number of dimensions and metric):

Definition

$$\delta F[\varphi] = F[\varphi + \delta\varphi] - F[\varphi] \equiv \int dx \delta\varphi(x) \frac{\delta F[\varphi]}{\delta\varphi(x)}. \tag{1.30}$$

Let's see how this definition works for some particular examples.

Example 1: $F[\varphi] = \varphi(x)$ identity.

$$F[\ ] : \begin{cases} \langle\varphi\rangle \to \langle\varphi\rangle \\ \varphi(x) \to \varphi(x), \\ \varphi(x) + \delta\varphi(x) \to \varphi(x) + \delta\varphi(x). \end{cases} \tag{1.31}$$

$$\delta F[\varphi] = F[\varphi + \delta\varphi] - F[\varphi] = \delta\varphi(x) \overset{\forall\delta\varphi}{=} \int dy \delta\varphi(y) \frac{\delta F[\varphi]}{\delta\varphi(y)} \Longrightarrow \boxed{\frac{\delta\varphi(x)}{\delta\varphi(y)} = \delta(x - y).} \tag{1.32}$$

[3]This condition can be relaxed if such terms are treated as perturbations or if generalizations of the standard Hamiltonian formalism were considered.

<u>*Example 2*</u>: $F[\varphi] = \partial_i \varphi(x) = \frac{\partial}{\partial x^i}\varphi(x)$ derivative.

$$F[\ ] : \begin{cases} \langle\varphi\rangle \to \langle\varphi\rangle \\ \varphi(x) \to \partial_i \varphi(x) \\ \varphi(x) + \delta\varphi \to \partial_i \varphi(x) + \partial_i \delta\varphi(x) \end{cases} \tag{1.33}$$

$$\delta F[\varphi] = \partial_i \delta\varphi(x) = \int dy \delta\varphi(y) \frac{\delta F[\varphi]}{\delta\varphi(y)} \,, \tag{1.34}$$

so (recall that $\delta\varphi(x)$ is arbitrary),

$$\frac{\partial}{\partial x_i}\delta\varphi(x) = \frac{\partial}{\partial x_i}\int dy \delta\varphi(y)\delta(x-y) = \int dy \delta\varphi(y)\left(\frac{\partial}{\partial x_i}\delta(x-y)\right), \tag{1.35}$$

$$\|$$

$$\int dy \delta\varphi(y) \frac{\delta(\partial_i^x \varphi(x))}{\delta\varphi(y)} \Rightarrow \boxed{\frac{\delta}{\delta\varphi(y)}\partial_i^x \varphi(x) = \frac{\partial}{\partial x^i}\delta(x-y) = -\frac{\partial}{\partial y^i}\delta(x-y).} \tag{1.36}$$

The last equalities should be carefully understood within distribution theory.

<u>*Example 3*</u>

$$\frac{\delta}{\delta\varphi(y)}\left[\partial_{i,\dots,i_k}\varphi(x)\right] = (-i)^k \frac{\partial^k}{\partial y^i, \dots, \partial y^{i_k}}\delta(x-y). \tag{1.37}$$

<u>*Example 4*</u>

- a constant: $\frac{\delta a}{\delta\varphi(x)} = 0$; $\frac{\delta}{\delta\varphi(x)}[aF + bG] = a\frac{\delta F}{\delta\varphi(x)} + b\frac{\delta G}{\delta\varphi(x)}$.
- $\frac{\delta}{\delta\varphi(x)}\left(F[\varphi]G[\varphi]\right) = \frac{\delta F[\varphi]}{\delta\varphi(x)}G[\varphi] + F[\varphi]\frac{\delta G[\varphi]}{\delta\varphi(x)}$.

Action

The action can be written as follows

$$S[\phi] = \int_{t_1}^{t_2} dt L([\phi], [\dot{\phi}]; t). \tag{1.38}$$

For simplicity, we omit the time integration limits in the following. We also omit any reference to the volume V where the Lagrangian is defined. By default, we set $V = \mathbb{R}^d$.

- Minimal action

  It is determined from the following equation (recall that $D = 1 + d =$ time + 'space' dimensions)

$$\frac{\delta S}{\delta\phi(x)} = 0, \quad \text{with } \delta S = \int d^D x \delta\phi(x) \frac{\delta S}{\delta\phi(x)} \,. \tag{1.39}$$

$$\begin{aligned}\delta S &= S[\phi+\delta\phi]-S[\phi]=\int dt\left[L([\phi+\delta\phi],[\dot\phi+\delta\dot\phi];t)-L([\phi],[\dot\phi];t)\right]\\ &= \int dt\int d^d\mathbf{x}\left[\delta\phi(t,\mathbf{x})\frac{\delta L}{\delta\phi(t,\mathbf{x})}+\delta\dot\phi(t,\mathbf{x})\frac{\delta L}{\delta\dot\phi(t,\mathbf{x})}\right]\end{aligned}$$

Definition of functional variation: variation over all independent fields

$$= \int dt\int d^d\mathbf{x}\left[\delta\phi\frac{\delta L}{\delta\phi}+\frac{d}{dt}\left(\delta\phi\frac{\delta L}{\delta\phi}\right)-\delta\phi\frac{d}{dt}\left(\frac{\delta L}{\delta\dot\phi}\right)\right], \tag{1.40}$$

This term is zero because
$\delta\phi(t_2,\mathbf{x})=\delta\phi(t_1,\mathbf{x})=0\ \forall\mathbf{x}\ \in V$

where $\delta\dot\phi=\frac{d}{dt}\delta\phi$. So, we have

$$0=\delta S=\int d^Dx\,\delta\phi\left[\frac{\delta L}{\delta\phi(t,\mathbf{x})}-\frac{d}{dt}\left(\frac{\delta L}{\delta\dot\phi(t,\mathbf{x})}\right)\right]\quad\forall\,\delta\phi$$

$$\Rightarrow \boxed{\frac{\delta L}{\delta\phi(t,\mathbf{x})}-\frac{d}{dt}\left(\frac{\delta L}{\delta\dot\phi(t,\mathbf{x})}\right)=0\,.} \tag{1.41}$$

This is the E-L EoM in its most general form if L only depends on, at most, first-order time derivatives of the fields.

In Eq. (1.41), $\frac{d}{dt}$ stands for the total time derivative. Its explicit form is given in the following exercise.

Exercise

Prove that

$$\frac{d}{dt}\frac{\delta L}{\delta\phi(t,\mathbf{x})}=\left[\int d^d\mathbf{y}\dot\phi(t,\mathbf{y})\frac{\delta}{\delta\phi(t,\mathbf{y})}+\ddot\phi(t,\mathbf{y})\frac{\delta}{\delta\dot\phi(t,\mathbf{y})}+\frac{\partial}{\partial t}\right]\frac{\delta L}{\delta\phi(t,\mathbf{x})}. \tag{1.42}$$

Hint. We can define

$$\frac{\delta L}{\delta\phi(t,\mathbf{x})}([\phi(t)],[\dot\phi(t)];t)\equiv f(t)\,, \tag{1.43}$$

and use the definition of derivatives

$$\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}\Longrightarrow\begin{cases}\delta\phi=\Delta t\ \dot\phi(t;\mathbf{y})\\ \delta\dot\phi=\Delta t\ \ddot\phi(t;\mathbf{y})\end{cases}. \tag{1.44}$$

Functional Densities

Actions and Lagrangians are typically expressed in terms of densities. We take the following expression as an example for an action of a relativistic theory:[4]

$$S[\phi] = \int d^D x \mathcal{L}(\phi(x), \phi_{/\mu}(x), \phi_{/\mu\nu}(x); x) \,. \tag{1.46}$$

We now want to determine how the E-L EoM looks like. The variation of the action reads

$$\begin{aligned}
\delta S &= \int d^D x \,[\, \mathcal{L}(\phi + \delta\phi, \phi_{/\mu} + \delta\phi_{/\mu}, \phi_{/\mu\nu} + \delta\phi_{/\mu\nu}; x) - \mathcal{L}(\phi, \phi_{/\mu}, \phi_{/\nu\mu}; x)] \\
&= \int d^D x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi_{/\mu}} \delta\phi_{/\mu} + \frac{\partial \mathcal{L}}{\partial \phi_{/\mu\nu}} \delta\phi_{/\mu\nu} \right] + \ldots \\
&= \int d^D x \; \delta\phi \left[\frac{\partial \mathcal{L}}{\partial \phi} - \left(\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{/\mu}} \right) + \left(\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial \phi_{/\mu\nu}} \right) \right] \quad \forall \delta\phi \,.
\end{aligned} \tag{1.47}$$

Note that the derivatives appearing in the second equality are the standard derivatives for functions, since the Lagrangian density is a function (not a functional) of the field variables. Also note that there is no variation of x. Only the fields are varied, since x is a label, not a variable. In the third equality, we have performed integration by parts and fixed $\delta\phi$ and its derivatives to be zero at the time and spatial boundaries. This eliminated total derivative terms such as

$$\int d^D x \left(\frac{\partial}{\partial x^\mu} \delta\phi \frac{\partial \mathcal{L}}{\partial \phi_{/\mu}} \right) \tag{1.48}$$

in the last equality. Overall, we obtain the following EoM:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \left(\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{/\mu}} \right) + \left(\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial \phi_{/\mu\nu}} \right) = 0 \,. \tag{1.49}$$

Note that this is not the standard form for the E-L EoM. The standard expression reads

$$\frac{\partial \mathcal{L}}{\partial \phi} - \left(\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{/\mu}} \right) = 0 \,. \tag{1.50}$$

This is the result obtained if the Lagrangian density has the following form

$$\mathcal{L}(\phi, \phi_{/\mu}; x) \,. \tag{1.51}$$

[4]Shortcut notation

$$\phi_{/\mu} \equiv \frac{\partial}{\partial x^\mu} \phi, \;\; \phi_{/\mu\nu} \equiv \frac{\partial}{\partial x^\mu \partial x^\nu} \phi, \;\; \delta\phi_{/\mu\nu} \equiv \frac{\partial}{\partial x^\mu \partial x^\nu} \delta\phi. \tag{1.45}$$

There are strong motivations for favoring Lagrangians of this form. First, we aim to include only first-order time derivatives, which allows us to transition to the Hamiltonian formalism and proceed with quantization. Furthermore, if we impose relativistic invariance, it links the number of time and spatial derivatives that can appear, leading to the general form presented in Eq. (1.51). However, this condition can be relaxed: higher-order time (and consequently space) derivatives may be included if treated within perturbation theory. Additionally, relativity enforces translational invariance, as we will discuss in Sec. 4.2.2, implying that the Lagrangian density must not depend explicitly on the x space-time coordinates. On the other hand, if we only require the presence of, at most, first-order time derivatives—without enforcing relativity—the Lagrangian may have this form

$$L([\phi],[\dot{\phi}];t) = \int d^d\mathbf{x}\mathcal{L}(\phi,\partial_i\phi,\partial_{ij}\phi,\ldots;\dot{\phi},\partial_i\dot{\phi},\partial_{ij}\dot{\phi},\ldots;t,\mathbf{x}) \tag{1.52}$$

with an arbitrary number of spatial derivatives.

Finally, let us emphasize that both Lagrangians, Eqs. (1.51) and (1.52), yield Eq. (1.41), as they only depend on first-order time derivatives at most. On the other hand, the explicit expression for the E-L EoM will differ once written in terms of the respective Lagrangian densities.

1.3.4 Hamiltonian Formalism

CM

We consider a system with N (fixed and small) degrees of freedom. We assume that its dynamics can be described by the following Lagrangian:

$$L(q_i,\dot{q}_i,t). \tag{1.53}$$

Instead of q and $\dot{q}$, the Hamiltonian formalism uses the canonical coordinates: q and p. Therefore, we replace $\dot{q}$ with p:

$$\dot{q}_i \to p_i = \frac{\partial L}{\partial \dot{q}_i} = f(q,\dot{q},t)\,. \tag{1.54}$$

It is important to note that we can only perform this change of variables if it is invertible. In other words, if the Jacobian is different from zero: $\left|\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_i}\right| \neq 0$. In the following discussion, we assume this condition is satisfied.

We can also define the Hamiltonian:

$$H(q_i,p_i,t) \equiv \sum_{i=1}^{n} p_i\dot{q}_i - L(q,\dot{q},t)\,, \tag{1.55}$$

where $\dot{q} = \dot{q}(q,p,t)$.

We can derive the EoM from this quantity using Hamilton's equations

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,, \tag{1.56}$$

where

$$\{F,G\} \equiv \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\,. \tag{1.57}$$

Of particular relevance to us is the particular case:

$$\{q_i, p_j\} = \delta_{ij}\,, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0\,, \tag{1.58}$$

since it leads to the canonical quantization condition:

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \qquad [\hat{q}_i, \hat{q}_i] = [\hat{p}_i, \hat{p}_j] = 0\,. \tag{1.59}$$

Note that in this equation $\hat{q}$ and $\hat{p}$ are operators acting on a suitable Hilbert space.

CFT

Let us now turn to CFT (N large and arbitrary). As before, we can also move to canonical variables by defining the momentum as:

$$\pi(t, \mathbf{x}) \equiv \frac{\delta L}{\delta\dot{\phi}(\mathbf{x}, t)}\,, \tag{1.60}$$

and the Hamiltonian as

$$H\left([\phi], [\pi], t\right) \equiv \int d^d\mathbf{x}\, \dot{\phi}(\mathbf{x}, t)\pi(\mathbf{x}, t) - L\left([\phi], [\dot{\phi}], t\right)\,, \tag{1.61}$$

where $\dot{\phi} \equiv \dot{\phi}\left([\phi], [\pi]; t\right)$.

To move to canonical coordinates (and back) we need the Jacobian of the canonical transformation to be different from zero:

$$\left|\frac{\delta^2 L}{\delta\dot{\phi}(\mathbf{x}', t)\delta\dot{\phi}(\mathbf{x}, t)}\right| \neq 0, \tag{1.62}$$

so that the inverse of the canonical transformation exists (this is a problem in realistic physical cases, especially in those involving photons).

Hamilton's equation for a general functional $F = F([\phi], [\pi]; t)$ now reads

$$D_t F = \{F, H\} + \frac{\partial F}{\partial t}, \tag{1.63}$$

where D_t is the total time derivative of the functional, and we have used the following definition:

Definition of Poisson's Bracket

$$\{F, G\} = \int d^d\mathbf{x} \left(\frac{\delta F}{\delta\phi(t,\mathbf{x})} \frac{\delta G}{\delta\pi(t,\mathbf{x})} - \frac{\delta G}{\delta\phi(t,\mathbf{x})} \frac{\delta F}{\delta\pi(t,\mathbf{x})} \right). \tag{1.64}$$

This equation implies that

$$\{\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')\} = \delta^{(d)}(\mathbf{x} - \mathbf{x}')\,, \tag{1.65}$$

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = [\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = 0\,. \tag{1.66}$$

If the Lagrangian can be written in terms of a Lagrangian density

$$L\left([\phi], [\dot\phi]; t\right) = \int d^d\mathbf{x}\, \mathcal{L}\left(\phi(x), \phi_{/\mu}(x); t, \mathbf{x}\right)\,, \tag{1.67}$$

the following simplifications occur:

$$\pi(t,\mathbf{x}) \equiv \frac{\delta L}{\delta\dot\phi(t,\mathbf{x})} = \frac{\partial\mathcal{L}}{\partial\dot\phi}\,, \tag{1.68}$$

and

$$H\left([\phi], [\pi], t\right) \equiv \int d^d\mathbf{x} \left(\dot\phi(x)\pi(x) - \mathcal{L}\left(\phi(x), \dot\phi(x); x\right)\right) = \int d^d\mathbf{x}\, \mathcal{H}, \tag{1.69}$$

where

$$\mathcal{H}\left(\phi(x), \pi(x); x\right) = \dot\phi(x)\pi(x) - \mathcal{L}\left(\phi(x), \dot\phi(x); t, \mathbf{x}\right). \tag{1.70}$$

Quantization

Working with the canonical formalism allows us to perform *canonical quantization*, which means promoting the classical Poisson brackets of the canonical coordinates to quantum commutation relations:

$$\boxed{\begin{aligned} \left[\hat\phi(t,\mathbf{x}), \hat\pi(t,\mathbf{x}')\right] &= i\hbar\delta^{(d)}(\mathbf{x} - \mathbf{x}')\,, \\ \left[\hat\phi(t,\mathbf{x}), \hat\phi(t,\mathbf{x}')\right] &= [\hat\pi(t,\mathbf{x}), \hat\pi(t,\mathbf{x}')] = 0\,. \end{aligned}} \tag{1.71}$$

We conclude with the following remarks (some of which will also appear later but are worth mentioning at this stage).

Remarks

- Unless stated otherwise, we will set $\hbar = 1$ throughout the book.
- Note that, while $\hat{p}$ can be interpreted as the generator of spatial translations in quantum mechanics, $\hat{\pi}(t, \mathbf{x})$ does not play the same role in QFTs (see Exercise 8 in Sec. 1.3.5).
- In QFT the fields are operators. Therefore, fields do not have probabilistic interpretation.
- Related to the previous point, there is no first and second quantization. There is only one quantization (albeit with an arbitrary number of degrees of freedom) with operators acting on elements of a Hilbert space (as in standard quantum mechanics courses) generating standard matrix elements: $\langle \quad |\hat{O}(\mathbf{x})| \quad \rangle$.
- Treating space and time on an equal footing will naturally lead us to work in the Heisenberg picture.

1.3.5 Exercises

Some of these exercises require application of Noether's theorem, which will be presented in Sec. 4.2.1.

1. Compute
$$\frac{\delta F[\psi]G[\psi]}{\delta\psi(x)} .$$

2. Compute
$$\frac{\delta F[\psi]}{\delta\psi(\mathbf{x})} ,$$
where
$$F[\psi] = \int d^d\mathbf{x}\psi(\mathbf{x})e^{(\boldsymbol{\nabla}\psi(\mathbf{x}))^2} .$$
Compute
$$\frac{\delta F[\psi]}{\delta\psi(\mathbf{x})} ,$$
where
$$F[\psi] = \int d^d\mathbf{x}\ \psi^4(\mathbf{x})\frac{1}{1+(\boldsymbol{\nabla}\psi(\mathbf{x}))^2} .$$

3. Using the principle of minimal action, find the EoM for the Lagrangian
$$L = \int d^d\mathbf{x} \left\{\frac{\mu}{2}(\partial_t\psi(t,\mathbf{x}))^2 - \frac{y}{2}(\boldsymbol{\nabla}\psi(t,\mathbf{x}))^2\right\} .$$

4. Using the principle of minimal action, find the EoM for the Lagrangian

$$
\begin{aligned}
L &= \int d^d\mathbf{x}\Big\{ i\psi^*(t,\mathbf{x})(\partial_t\psi(t,\mathbf{x})) - \frac{1}{2m}(\boldsymbol{\nabla}\psi^*(t,\mathbf{x}))(\boldsymbol{\nabla}\psi(t,\mathbf{x})) \\
&\quad -V(\mathbf{x})\psi^*(t,\mathbf{x})\psi(t,\mathbf{x})\Big\}\,. \qquad (1.72)
\end{aligned}
$$

   Also obtain, if possible, the four-vector P^μ using Noether's theorem.

5. a) Using Noether's theorem obtain the four-vector P^μ for the Klein-Gordon real free field theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\,;$$

   b) Determine the E-L and Hamilton's EoM. Also determine H.

6. a) Using Noether's theorem obtain the four-vector P^μ for the KG complex free field theory:

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi\,;$$

   b) Determine the E-L and Hamilton's EoM. Also determine H.

7. Study the continuous and discrete symmetries, and obtain the associated charges (using Noether's theorem when possible), of the Lagrangians in exercises 5 and 6.

8. Shift symmetry. Consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2\,,$$

   and the transformation: $\phi(x) \to \phi(x) + \lambda$. Find the generator of this symmetry.

1.4 Poincaré Group*

1.4.1 Group Definitions

1. A set G is a group if it has an operation "$\cdot$" with the following properties:
   - Closure: If $g_1, g_2 \in G$, then $g_1 \cdot g_2 \in G$.
   - Associativity: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.
   - Identity: There exists $e \in G$ such that $e \cdot g = g \cdot e = g \ \ \forall\, g \in G$.
   - Inverse: For every $g \in G$, there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.
2. A group is non-abelian if $g_1 \cdot g_2 \neq g_2 \cdot g_1$ in general, and abelian if $g_1 \cdot g_2 = g_2 \cdot g_1 \ \ \forall\, g_1, g_2 \in G$.
3. $O(N)$: The set of real orthogonal matrices A of dimension $N \times N$ such that $A \cdot A^T = A^T \cdot A = I$.
4. $SO(N)$: The set of real orthogonal matrices A of dimension $N \times N$ with determinant 1: $A \cdot A^T = A^T \cdot A = I$, and $\det(A) = 1$.
5. $so(N)$: The set of real antisymmetric matrices A of dimension $N \times N$: $A = -A^T$.
6. $U(N)$: The set of unitary complex matrices U of dimension $N \times N$.
7. $SU(N)$: The set of unitary complex matrices U of dimension $N \times N$ with determinant 1: $\det(U) = 1$.
8. $su(N)$: The set of complex antihermitian matrices A of dimension $N \times N$ with trace zero: $A = -A^\dagger$, and $\mathrm{Tr}[A] = 0$.
9. Associative Algebra A: A vector space over $\mathbb{R}$ or $\mathbb{C}$ with an internal product

$$\begin{aligned} (A, A) &\longrightarrow A \\ (a, b) &\longrightarrow a \cdot b \end{aligned}$$

   satisfying:

   - Distributivity:

$$(\alpha a + \beta c) \cdot b = \alpha(a \cdot b) + \beta(c \cdot b) \tag{1.73}$$
$$a \cdot (\alpha b + \beta c) = \alpha(a \cdot b) + \beta(a \cdot c) \tag{1.74}$$

   - Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

10. **Lie Algebra** L: A vector space over $\mathbb{R}$ or $\mathbb{C}$ with an internal product called the Lie bracket $[A, B]$:

$$\begin{array}{rcl}(L, L) & \longrightarrow & L \\ (A, B) & \longrightarrow & [A, B]\end{array}$$

satisfying:

- Antisymmetry: $[A, B] = -[B, A]$
- Distributivity: $[A + B, C] = [A, C] + [B, C]$
- Jacobi Identity: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

11. Remark. Elements of $SU(N)$ and $su(N)$ act as linear operators on a complex vector space of dimension N.
12. Remark. Elements of $SO(N)$ and $so(N)$ act as linear operators on a real vector space of dimension N.
13. Remark. $so(N)$ and $su(N)$ are Lie algebras over $\mathbb{R}$.
14. A **representation of** L is any mapping

$$\begin{array}{rcl}T : L & \longrightarrow & H \\ X & \longrightarrow & T(X)\end{array}$$

where H is a set of linear operators acting on a vector space-V over $\mathbb{R}$ or $\mathbb{C}$ such that:

- $T(A + B) = T(A) + T(B)$
- $T([A, B]) = [T(A), T(B)]$

15. **Defining** Representation: Defined as the representation where $H = L$ and $T(X) = X$.
16. Remark. Every representation is determined by the behavior of the basis $\tilde{\Gamma}^a$. We consider Lie algebras over $\mathbb{R}$ with structure constants $f^{abc} \in \mathbb{R}$:

$$[\tilde{\Gamma}^a, \tilde{\Gamma}^b] = -f^{abc}\tilde{\Gamma}^c. \tag{1.75}$$

17. **Equivalent** Representations: Two representations $\tilde{\Gamma}^a$ and $\tilde{\Theta}^a$ are equivalent if $\exists\ \xi\ /\ \xi\tilde{\Theta}^a\xi^{-1} = \tilde{\Gamma}^a$. This implies that

$$[\tilde{\Gamma}^a, \tilde{\Gamma}^b] = -f^{abc}\tilde{\Gamma}^c \Rightarrow [\tilde{\Theta}^a, \tilde{\Theta}^b] = -f^{abc}\tilde{\Theta}^c\,.$$

18. **Complex Conjugate** Representation: Given a representation $\tilde{\Gamma}^a$, its complex conjugate $\tilde{\Gamma}^{a*}$ satisfies $[\tilde{\Gamma}^{a*}, \tilde{\Gamma}^{b*}] = -f^{abc}\tilde{\Gamma}^{c*}$.

19. **Real** Representation: A representation is real if it is equivalent to its complex conjugate.

20. **Adjoint** Representation: Defined as $T(X) \equiv \mathrm{ad}_X$, where

$$\begin{array}{rcl} \mathrm{ad}_X : L & \longrightarrow & L \\ Y & \longrightarrow & \mathrm{ad}_X(Y) \equiv [X, Y] \end{array}$$

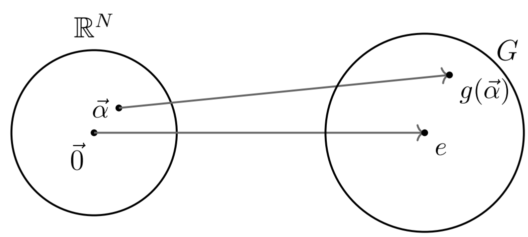
21. **Unitary** Representation: A representation T is unitary if $T^\dagger(A) = -T(A)$ $\forall\ A \in L$.

22. **Reducible** Representation: A representation T is reducible if there exists a subspace $U \subset V$ (with $U \neq V$) / $T(A)U \subset U\ \forall\ A \in L$.

23. **Irreducible** Representation: A representation T is irreducible if $\nexists\ U \subset V$ (with $U \neq V$) / $T(A)U \subset U\ \forall\ A \in L$.

24. **Completely Reducible** Representation: A representation T is completely reducible if it is reducible and $T(A)U \subset U\ \forall A \in L \Rightarrow T(A)\bar{U} \subset \bar{U}$ $\forall A \in L$ and $V = U \oplus \bar{U}$.

25. **Lie Groups**. They are defined as groups where it is possible to tag all its elements with elements of a domain of $\mathbb{R}^N$. Moreover, these mappings between elements of the group and the N dimensional manifold are performed via analytic functions.



We define (up to a sign) the **generators**, L^i, following the usual convention in the physics community:

$$-\, iL^i = \left.\frac{\partial g(\vec{\alpha})}{\partial \alpha^i}\right|_{\vec{\alpha}=0} . \tag{1.76}$$

Note that with this definition, iL^i are elements of the Lie algebra. They yield a basis of the Lie algebra in $\mathbb{R}^N$. For (simple) connected Lie groups, all elements of the group can be obtained via exponentiation of its generators. Therefore, in most cases, studying the Lie algebra will prove sufficient for our purposes.

1.4.2 Lorentz Group and Lie Algebra

The Lorentz group (boosts and rotations) can be defined by the condition that the Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}, \tag{1.77}$$

leave the Minkowski metric invariant:

$$x'^{\mu} x'_{\mu} = x'^{\mu} x'^{\nu} g_{\mu\nu} = \Lambda^{\mu}_{\ \sigma} x^{\sigma} \Lambda^{\nu}_{\ \rho} x^{\rho} g_{\mu\nu} = x^{\sigma} x^{\rho} g_{\sigma\rho} \quad \Longrightarrow \quad \Lambda^{\mu}_{\ \sigma} \Lambda^{\nu}_{\ \rho} g_{\mu\nu} = g_{\sigma\rho} \,. \tag{1.78}$$

We restrict to the subgroup

$$O(1,3)\big|_{\det(\Lambda)=1,\, \Lambda^{0}_{\ 0}>0} \longrightarrow SO_o(1,3), \tag{1.79}$$

where $SO_o(1,3)$ is the subgroup of the Lorentz group continuously connected to the identity (the proper orthochronous Lorentz group).

We can write any Lorentz transformation $\Lambda \in SO_o(1,3)$ as

$$\Lambda = BR = e^{-i\mathbf{\Phi}\cdot\mathbf{K}} e^{-i\mathbf{\Theta}\cdot\mathbf{M}}, \tag{1.80}$$

where R represents a rotation and B a boost. The parameters $\mathbf{\Phi}$ are normal coordinates of the group. In physical terms, they are rapidity coordinates, and are unbounded. On the other hand, the $\mathbf{\Theta}$ coordinates, the angles of the rotation, are bounded. The generators of the rotations read $((M_x, M_y, M_z) \equiv (M^1, M^2, M^3))$

$$M_x = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, M_y = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M_z = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.81}$$

In components they can be written in a compact way as follows: $(M^k)^{ij} = -i\epsilon^{kij}$.

For boosts, the generators read $((K_x, K_y, K_z) \equiv (K^1, K^2, K^3))$:

$$K_x = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_y = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_z = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.82}$$

The Lie algebra of $SO(1,3)$ reads

$$[M^i, M^j] = i\epsilon^{ijk} M^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} M^k, \qquad [M^i, K^j] = i\epsilon^{ijk} K^k. \tag{1.83}$$

The Lorentz generators can be compactly written as a single tensor, $J^{\mu\nu}$, with the following structure

$$J^{\mu\nu} = \begin{cases} J^{ij} = -J^{ji} \equiv \epsilon^{ijk}\mathbf{M}^k \\ J^{i0} = -J^{0i} \equiv -\mathbf{K}^i \end{cases} .$$

These 4×4 matrices can be written in a compact form in terms of their components as follows:

$$(J^{\mu\nu})_{\alpha\beta} = -i(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha) .$$

The Lorentz matrix can then be written as follows (we leave it to the reader to identify the antisymmetric tensor $w_{\mu\nu}$ with the parameters $\mathbf{\Phi}$ and $\mathbf{\Theta}$):

$$(\Lambda)^\rho{}_\alpha = \left(e^{\frac{i}{2} w_{\mu\nu} J^{\mu\nu}}\right)^\rho{}_\alpha \simeq \delta^\rho{}_\alpha + w^\rho{}_\alpha ,$$

and the Lorentz Lie algebra in terms of $J^{\mu\nu}$ reads

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} + g^{\mu\sigma} J^{\nu\rho} - g^{\nu\sigma} J^{\mu\rho}). \qquad (1.84)$$

1.4.3 Action on Fields

The Lorentz Lie algebra can also be derived from alternative representations. Particularly useful to us is the one obtained by applying Lorentz transformations to scalar fields. We define the general transformation as follows:

$$\phi(x) \to \phi(\Lambda^{-1}x) \equiv T(\Lambda)\phi(x) .$$

Under a rotation, a scalar field $\phi(x)$ transforms as

$$\phi(x) \to \phi(R^{-1}x) \simeq \phi(x + i\delta\varphi\, \hat{n} \cdot \mathbf{M}\, x) \simeq \phi(x) - i\delta\varphi\, \hat{n} \cdot T(\mathbf{M})\, \phi(x), \qquad (1.85)$$

where $\phi(x)$ is a scalar function and $T(\mathbf{M}) = \mathbf{x} \times \mathbf{p}$, where $\mathbf{p} \equiv -i\mathbf{\nabla}$.

For a boost in the x-direction (standard boost) with velocity v, we have:

$$\begin{aligned} x' &= \gamma(x + vt), \qquad & y' &= y, \\ t' &= \gamma(t + vx), \qquad & z' &= z, \end{aligned}$$

which gives the infinitesimal transformation

$$\phi(B^{-1}x) \simeq \phi(x + i\delta v K_x x) \simeq \phi(x) - i\delta v T(K_x)\phi(x); \qquad T(K_x) = -i(t\partial_x + x\partial_t). \qquad (1.86)$$

and similarly in the other directions. Overall, the boost generators read

$$T(K_x) = -i(t\partial_x + x\partial_t) , \qquad T(K_y) = -i(t\partial_y + y\partial_t) , \qquad T(K_z) = -i(t\partial_z + z\partial_t) . \qquad (1.87)$$

In general, under a finite Lorentz transformation, we have

$$x' = \Lambda x \quad \Rightarrow \quad \phi(x') = e^{-\frac{i}{2}w_{\mu\nu}T(J^{\mu\nu})}\phi(x) \simeq \phi(x) - \frac{i}{2}\delta\omega_{\mu\nu}T(J^{\mu\nu})\phi(x)\,, \quad (1.88)$$

where

$$T(J^{\mu\nu}) = i(x^\mu\partial^\nu - x^\nu\partial^\mu). \quad (1.89)$$

1.4.4 Poincaré Group and Lie Algebra

The Poincaré group includes space-time translations in addition to Lorentz transformations:

$$x'^\mu = \Lambda^\mu_{\ \nu}x^\nu + a^\mu. \quad (1.90)$$

Thus, we introduce four additional generators P^μ for space-time translations.

Under space-time translations, a scalar field $\phi(x)$ transforms as

$$\phi(x) \to e^{i\delta a\cdot T(P)}\phi(x) = \phi(x - \delta a) = \phi(x) - \delta a_\mu \frac{\partial}{\partial x_\mu}\phi(x), \quad (1.91)$$

so that $T(P^\mu) \equiv P^\mu = i\partial^\mu$ are the generators of translations.

Lie algebra of the Poincaré Group

The complete Poincaré Lie algebra is obtained by adding

$$[P^\mu, P^\nu] = 0, \qquad [P^\mu, J^{\rho\sigma}] = i\left(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho\right). \quad (1.92)$$

to the commutator relations obtained in Eq. (1.84).

Generators Summary

$$\left.\begin{array}{ll} \text{3 rotations:} & J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \\ \text{3 boosts:} & K^i = J^{0i}, \\ \text{4 translations:} & P^\mu, \end{array}\right\} \Longrightarrow \text{basis of the Poincaré Lie algebra.} \quad (1.93)$$

It should be emphasized that the Lie algebra of the Poincaré group does not involve any $\hbar$.

1.4.5 Exercises

1. Compute $L_E \equiv e^{\theta\sigma}$ and $x' = e^{\theta\sigma}x$, where $x = (x_1, x_2, x_3, x_4)$ and $(\lambda^2 + \mu^2 + \nu^2 = 1)$

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & i\lambda \\ 0 & 0 & 0 & i\mu \\ 0 & 0 & 0 & i\nu \\ -i\lambda & -i\mu & -i\nu & 0 \end{pmatrix}.$$

2. Rewrite the result of the previous exercise after applying the transformation $x_4 = ict = ix_0$ and using the coordinates (x_0, x_1, x_2, x_3). Include rotations and obtain the 4×4 matrix generators of the group $SO(1,3)$. Obtain the associated Lie algebra by computing the commutation relations among the different generators of the symmetry group. Finally, try to express the result in a manifestly covariant form.

3. Determine the Lie algebra of the subgroup of the Lorentz transformations that leave the vector $k^\mu = (k, 0, 0, k)$ invariant (see Sec. 3.3.5).

4. We perform two boosts. The first one with velocity v_1 along the axis x_1. Then a second with velocity v_2 along the axis x_2. Prove that the result cannot be understood as a pure boost.

5. Consider the group SO(1,3) in its defining representation. Given $\epsilon^{\mu\nu\rho\sigma}$, the Levi-Civita tensor, completely anti-symmetric, and with $\epsilon^{0123} = 1$, determine the value of ϵ_{0123}.

6. Compute the determinant of a general Lorentz transformation.

7. Compute the following commutators:

$$[J_{\mu\nu}, J_{\alpha\beta}] \qquad [J_{\mu\nu}, P_\alpha]\,,$$

and

$$[J_{\mu\nu}, W_\alpha] \qquad [P_\mu, W_\alpha]\,,$$

where

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma\,. \tag{1.94}$$

2 Non-relativistic QFT

In this chapter, we discuss NR QFTs. These QFTs will help us to highlight aspects of QFTs that already appear in a NR setup.

2.1 NR Many-body Systems (bosons)

2.1.1 CFT Analysis

We start by working in CFT. Let us consider the following Lagrangian density in d (=3) spatial dimensions:

$$\mathcal{L} = \underbrace{i\varphi^*\dot{\varphi}}_{\frac{i}{2}[\varphi^*(\partial_0\varphi)-(\partial_0\varphi^*)\varphi]} - \frac{1}{2m}(\boldsymbol{\nabla}\varphi^*)(\boldsymbol{\nabla}\varphi) - V(\mathbf{x})\varphi^*\varphi; \quad S = \int d^D x\mathcal{L}, \tag{2.1}$$

where $\dot{\varphi} \equiv \frac{\partial\varphi}{\partial t}$, and we have shown two possible ways to write the first term. Both yield the same EoM, but only the one below is Hermitian/real. In either case, we can easily obtain the EoM:

$$\delta S = 0 \Rightarrow \text{E-L EoM} \Rightarrow i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\left(\boldsymbol{\nabla}^2\varphi\right) + V(\mathbf{x})\varphi\,. \tag{2.2}$$

Note that this is the Schrödinger equation. Even so, we do not assign any probabilistic interpretation to φ—as expected, since we are still working within the framework of CFT. More importantly, this statement will also remain valid after quantization, since φ will become an operator rather than a wave function.

We now move towards the Hamiltonian formalism

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = i\varphi^*; \quad \pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}^*} = 0 \Rightarrow \mathcal{H} = \pi\dot{\varphi} + \pi^*\dot{\varphi}^* - \mathcal{L}. \tag{2.3}$$

So,[1]

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{\nabla}\varphi^*)(\boldsymbol{\nabla}\varphi) + V(\mathbf{x})\varphi^*\varphi \Rightarrow H = \int d^d\mathbf{x}\varphi^*(\mathbf{x}) \underbrace{\left[-\frac{1}{2m}\boldsymbol{\nabla}^2 + V(\mathbf{x})\right]}_{\hat{h}_{\mathbf{x}}} \varphi(\mathbf{x})\,, \tag{2.4}$$

[1] It is interesting to note that one obtains the same expression for the Hamiltonian if one uses $\frac{i}{2}[\varphi^*(\partial_0\varphi) - (\partial_0\varphi^*)\varphi]$ instead of $i\varphi^*\dot{\varphi}$.

for which in the last equality we performed integration by parts and neglected the boundary contribution at spatial infinity. This leads to the appearance of $\hat{h}_{\mathbf{x}}$, which can be considered an operator acting on elements of the Hilbert space $\mathcal{L}^2_{\mathbb{R}^d}$. This establishes an important link to what is called first quantization. Note that, in general, $\hat{h}_{\mathbf{x}}\varphi \neq \lambda\varphi$, which means that φ is not an eigenstate of $\hat{h}$.

Change of Basis

To connect with the concept of particles, it is useful to change basis.

- The eigenstates of $\hat{h}_{\mathbf{x}}$:
$$\hat{h}\varphi_n(\mathbf{x}) = e_n\varphi_n(\mathbf{x})\,, \tag{2.5}$$
yield a basis $\{\varphi_n\}$ of $\mathcal{L}^2_{\mathbb{R}^d}$. We can do this for any Sturm-Liouville operator. If $V(\mathbf{x}) = 0$, the eigenfunctions are plane waves, $\varphi_n \sim e^{i\mathbf{p}_n\cdot\mathbf{x}}$, and n becomes a continuous variable when the volume is $V = \mathbb{R}^d$. In a finite volume, n is discrete.

- The wavefunction $\varphi(\mathbf{x})$ can then be expanded as:
$$\varphi(\mathbf{x}) = \sum_n a_n\varphi_n(\mathbf{x}), \quad \varphi^*_n(\mathbf{x}) = \sum_n a^*_n\varphi^*_n(\mathbf{x})\,. \tag{2.6}$$

- We choose the basis to be orthogonal:
$$\int d^d\mathbf{x}\,\varphi^*_m(\mathbf{x})\varphi_n(\mathbf{x}) = \delta_{mn}. \tag{2.7}$$
We can then obtain the expansion coefficients a_n by projecting the wave function to each eigenstate:
$$a_n = \int d^d\mathbf{x}\,\varphi^*_n(\mathbf{x})\varphi(\mathbf{x}). \tag{2.8}$$

Hamiltonian Representation

Where does this change of variable lead us in the Hamiltonian formalism?

We have
$$\hat{h}_{\mathbf{x}}\varphi(\mathbf{x}) = \sum_n \hat{h}_n a_n\varphi_n(\mathbf{x}) = \sum_n e_n a_n\varphi_n(\mathbf{x}). \tag{2.9}$$

Therefore,

Hamiltonian Expression

$$H = \int d^d\mathbf{x}\left(\sum_m \varphi^*_m(\mathbf{x})a^*_m\right)\left(\sum_n a_n e_n\varphi_n(\mathbf{x})\right) \stackrel{\delta_{nm}}{=} \sum_n e_n a^*_n a_n. \tag{2.10}$$

2.1.2 Quantization

We are now in the position to quantize the theory. We have two options.

Option A: Correspondence Principle (canonical quantization)
Using $\pi = i\varphi^\dagger$, we have

$$\{\varphi(\mathbf{x}), \pi(\mathbf{y})\} = \delta^{(d)}(\mathbf{x}-\mathbf{y}) \quad\longrightarrow\quad [\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\hbar\delta^{(d)}(\mathbf{x}-\mathbf{y}), \tag{2.11}$$

$$\{\varphi(\mathbf{x}), \varphi(\mathbf{y})\} = 0 \quad\longrightarrow\quad [\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{y})] = 0, \tag{2.12}$$

$$\{\pi(\mathbf{x}), \pi(\mathbf{y})\} = 0 \quad\longrightarrow\quad [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = 0\,, \tag{2.13}$$

where in the right-hand side we have operators. This is emphasized by introducing the $\widehat{\ }$ notation.

Note that the first equality can also be written as

$$\left[\hat{\varphi}(\mathbf{x}), \hat{\varphi}^\dagger(\mathbf{y})\right] = \delta^{(d)}(\mathbf{x}-\mathbf{y})\,. \tag{2.14}$$

The quantized field (operator) can be written in terms of the eigenstates of $\hat{h}_x$

$$\hat{\varphi}(\mathbf{x}) = \sum_n \hat{a}_n \varphi_n(\mathbf{x}). \tag{2.15}$$

Note that the a_n's have now become operators, since

$$\hat{a}_n = \int d^d\mathbf{x}\, \varphi_n^*(\mathbf{x})\hat{\varphi}(\mathbf{x})\,. \tag{2.16}$$

It is straightforward to show that their commutation relations are

$$[\hat{a}_n, \hat{a}_m] = [\hat{a}_n^\dagger, \hat{a}_m^\dagger] = 0, \;\; [\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}. \tag{2.17}$$

To illustrate, we prove the last equality:

$$\begin{aligned}
&\left[\int d^d\mathbf{x}\, \varphi_n^*(\mathbf{x})\hat{\varphi}(\mathbf{x}), \int d^d\mathbf{y}\, \varphi_m(\mathbf{y})\hat{\varphi}^\dagger(\mathbf{y})\right] \qquad (2.18)\\
&= \int d^d\mathbf{x}\, \varphi_n^*(\mathbf{x}) \int d^d\mathbf{y}\, \varphi_m^*(\mathbf{y}) \left[\hat{\varphi}(\mathbf{x}), \hat{\varphi}^\dagger(\mathbf{y})\right]\\
&= \int d^d\mathbf{x}\, \varphi_n^*(\mathbf{x}) \int d^d\mathbf{y}\, \varphi_m^*(\mathbf{y})\delta^{(d)}(\mathbf{x}-\mathbf{y}) = \int d^d\mathbf{x}\, \varphi_n^*(\mathbf{x})\varphi_m(\mathbf{x}) = \delta_{nm}.
\end{aligned}$$

Then, the Hamiltonian operator is given by

$$\hat{H} = \sum_n e_n \hat{a}_n^\dagger \hat{a}_n. \tag{2.19}$$

Conclusion. The commutation relations for the $\hat{a}$'s, Eq. (2.17), and Eq. (2.19), lead to the following interpretation:

We have an infinite number of decoupled oscillators with different frequencies.

We know how to quantize these: we create a vacuum state $|0\rangle$ for each oscillator and then generate new states by applying the creation operator as many times as necessary:

$$\hat{a}^\dagger |0\rangle \sim |1\rangle \ldots . \tag{2.20}$$

We elaborate on this in the following section.

2.1.3 Hilbert Space and Fock Space

Overall, the structure of the theory is the following:

- Hilbert space $\equiv$ Fock space $\equiv$ direct product of the Hilbert space of each oscillator (and we have an infinite number of them).
- Ground state $\equiv$ vacuum

$$|\text{vac}\rangle = |0\rangle = |(0,0,\ldots)\rangle = |0\rangle_1 |0\rangle_2 \ldots |0\rangle_n \ldots |0\rangle_\infty . \tag{2.21}$$

- Other states are generated by applying the creation operators to the vacuum

$$(\hat{a}_1^\dagger |0\rangle_1) |0_2\rangle |0_3\rangle \cdots = \hat{a}_1^\dagger |0\rangle = |(1,0,0,\ldots,0)\rangle , \tag{2.22}$$

$$(\hat{a_1}^\dagger)^2 |0_1\rangle |0_2\rangle (\hat{a_3}^\dagger)^4 |0_3\rangle \cdots = (\hat{a}_1^\dagger)^2 (\hat{a_3}^\dagger)^4 |0\rangle = |(2,0,4,0\ldots)\rangle . \tag{2.23}$$

- If we want them to be normalized, we include a prefactor. A general state looks like this

General Fock State

$$|(N_1, N_2, \ldots, N_j, \ldots)\rangle = \frac{1}{\sqrt{N_1! N_2! \ldots N_j!}} (\hat{a}_1^\dagger)^{N_1} (\hat{a}_2^\dagger)^{N_2} \ldots (\hat{a}_j^\dagger)^{N_j} |0\rangle . \tag{2.24}$$

- These states are eigenstates of $\hat{H}$

$$\hat{H}\hat{a}_n^\dagger |0\rangle = \sum_m e_m \hat{a}_m^\dagger \hat{a}_m \hat{a}_n^\dagger |0\rangle = e_n \hat{a}_n^\dagger |0\rangle \,, \tag{2.25}$$

$$\hat{H}\,|(N_1,\ldots,N_j,\ldots)\rangle = \left(\sum_{j=1}^{\infty} N_j e_j\right) |(N_1,N_2,\ldots,N_j,\ldots)\rangle \,. \tag{2.26}$$

We have N_j particles with energy e_j.

Alternative quantization conditions

The previous construction of the Hilbert space using creation and annihilation operators, along with the associated vacuum, leads to the following question: What is the quantization condition? The one on the left-hand side or the one on the right-hand side below?

$$\begin{array}{c} \text{Option A} \\ \left[\hat{\varphi},\hat{\varphi}^\dagger\right] = \delta \\ \left[\hat{\varphi},\hat{\varphi}\right] = \left[\hat{\varphi}^\dagger,\hat{\varphi}^\dagger\right] = 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \text{Option B} \\ \left[\hat{a}_n,\hat{a}_m^\dagger\right] = \delta_{nm}, \\ \left[\hat{a}_n,\hat{a}_m\right] = \left[\hat{a}_n^\dagger,\hat{a}_m^\dagger\right] = 0. \end{array} \tag{2.27}$$

Both options appear to be valid prescriptions.[2] At this point, it is worth emphasizing that the right-hand side quantization conditions do not bear any $\hbar$ at all. However, it is precisely for those quantization conditions that the word *quantum* takes its full meaning, as the operators $\hat{a}_n^\dagger$ and $\hat{a}_n$ indicate the creation and annihilation of quanta with quantum numbers n, where n is directly linked to observable quantities. This raises the question of whether introducing $\hbar$ is necessary at all. In fact, with a proper rescaling of the fields, one could eliminate $\hbar$ from the equations entirely and describe the system purely in terms of frequencies, wavelengths, or the number of quanta present in a given state.

2.1.4 Evolution in Time

Once we have characterized the Hilbert space, we can implement the time evolution of its states using the

Time Evolution Operator

$$\hat{U} = e^{-it\hat{H}}. \tag{2.28}$$

Pictures

In Quantum Mechanics courses, it is standard to work in the Schrödinger picture where time evolution is incorporated into the states of the Hilbert space. For QFTs, however, it is more convenient to work in the Heisenberg picture, where

[2] In both cases, we also assume that the Hamiltonian is a positive semidefinite operator.

the time evolution is incorporated into the operators rather than into the states. The reason is that

$$\hat{x}_i(t) \to \hat{\phi}(t, \mathbf{x}), \tag{2.29}$$

so now time and space are continuous variables and it is convenient to treat them in a more symmetric way. This favours the Heisenberg picture. Either way, obviously, matrix elements remain invariant:

$${}_S \langle\psi| \hat{O}_S |\psi\rangle_S = {}_H \langle\psi| \hat{O}_H |\psi\rangle_H \,, \quad \hat{U}^\dagger \hat{O}_S \hat{U} = \hat{O}_H, \quad |\ \rangle_H = \hat{U}^\dagger |\ \rangle_S \,. \tag{2.30}$$

We set both pictures to be equal at a fixed time t_0, which we typically set to zero: $t_0 = 0$.

In summary, we have

- Schrödinger: $|\Psi(t)\rangle \,, \hat{O}(\not{t})$.
- Heisenberg: $|\Psi(\not{t})\rangle \,, \hat{O}(t)$.

We return to this discussion in Ch. 5 on relativistic theories.

Time Evolution of the Creation and Annihilation Operators

The time evolution of the creation operator in the Heisenberg picture is given by

$$\hat{a}_n^\dagger(t) = \hat{U}^\dagger \hat{a}_n^\dagger(0) \hat{U} = e^{i\hat{H}t} \hat{a}_n^\dagger e^{-i\hat{H}t}. \tag{2.31}$$

To solve this, we use the differential equation (note that $[\hat{H}, \hat{U}] = 0$)

$$\begin{aligned} \frac{d}{dt}\hat{a}_n^\dagger(t) &= -i\left[\hat{a}_n^\dagger(t), \hat{H}\right] = -i\left[\hat{U}^\dagger \hat{a}_n^\dagger \hat{U}, \hat{H}\right] = -i\hat{U}^\dagger \left[\hat{a}_n^\dagger, \hat{H}\right] \hat{U} \\ &= -i\hat{U}^\dagger \left[\hat{a}_n^\dagger, \sum_m e_m \hat{a}_m^\dagger \hat{a}_m\right] \hat{U} = \hat{U}^\dagger \sum_m e_m \hat{a}_m^\dagger [\hat{a}_n^\dagger, \hat{a}_m](-i)\hat{U} \\ &= ie_n \hat{U}^\dagger \hat{a}_n^\dagger(0) \hat{U} = ie_n \hat{a}_n^\dagger(t) \Rightarrow \boxed{\hat{a}_n^\dagger(t) = \hat{a}_n^\dagger e^{ie_n t}, \quad \hat{a}_n(t) = \hat{a}_n e^{-ie_n t}\,.} \end{aligned} \tag{2.32}$$

where we have used $\hat{a}_n^\dagger(0) = \hat{a}$ and the following identity

$$[AB, C] = A[B, C] + [A, C]B. \tag{2.33}$$

Time Evolution of the Field Operator

$$\hat{\varphi}^\dagger(t, \mathbf{x}) = \hat{U}^\dagger \hat{\varphi}^\dagger(t, \mathbf{x}) \hat{U} = \hat{U}^\dagger \sum_n \hat{a}_n^\dagger \varphi_n^*(\mathbf{x}) \hat{U} = \boxed{\sum_n \hat{a}_n^\dagger \varphi_n^*(\mathbf{x}) e^{ie_n t} = \hat{\varphi}^\dagger(t, \mathbf{x})\,.} \tag{2.34}$$

Note that the canonical commutation relations are preserved after time evolution:

$$\hat{\varphi}(t, \mathbf{x}) \hat{\varphi}^\dagger(t, \mathbf{y}) - \hat{\varphi}^\dagger(t, \mathbf{y}) \hat{\varphi}(t, \mathbf{x}) = \delta^{(d)}(\mathbf{y} - \mathbf{x}) \quad \forall\, t, \tag{2.35}$$

2.1.5 Localizability and Particle Interpretation of the Theory

Number Operator

The total number of particles in the system is measured by the number operator

$$\hat{N} \equiv \sum_n \hat{a}_n^\dagger \hat{a}_n \longrightarrow \hat{N}\,|(N_1, N_2, \ldots, N_j, \ldots)\rangle = \sum_i N_i\,|(N_1, N_2, \ldots, N_j, \ldots)\rangle\,, \tag{2.36}$$

whereas $\hat{N}_i \equiv \hat{a}_i^\dagger \hat{a}_i$ measures the number of particles with energy e_i.

Since the particle number operator is conserved, the following equalities hold

$$\hat{N}(t) = \hat{N}(0), \quad \hat{N}_i(t) = \hat{N}_i(0). \tag{2.37}$$

and its commutation relations remain invariant under time translations.

We now rewrite $\hat{N}$ in terms of the position field operators

Exercise

Show that

$$\hat{N} = \int d^d\mathbf{x}\,\hat{\varphi}^\dagger(t, \mathbf{x})\hat{\varphi}(t, \mathbf{x}) \quad \forall\, t\,. \tag{2.38}$$

This opens a window to look for observables that measure the number of particles in a finite volume. We consider the following operator

$$\hat{N}_V(t) = \int_V d^d\mathbf{x}\,\hat{\varphi}^\dagger(t, \mathbf{x})\hat{\varphi}(t, \mathbf{x}), \tag{2.39}$$

and see how it acts on the following state

$$|\mathbf{x}; t\rangle \equiv \hat{\varphi}^\dagger(t, \mathbf{x})\,|0\rangle\,. \tag{2.40}$$

Exercise

$$\begin{aligned}\hat{N}_V(t)\hat{\varphi}^\dagger(t,\mathbf{x})\,|0\rangle &= \left[\hat{N}_V(t),\hat{\varphi}^\dagger(t,\mathbf{x})\right]|0\rangle + \hat{\varphi}^\dagger(t,\mathbf{x})\hat{N}_V(t)\,|0\rangle \\ &= \left[\hat{N}_V(t),\hat{\varphi}^\dagger(t,\mathbf{x})\right]|0\rangle = \begin{cases}1\,|\mathbf{x};t\rangle\,, & \mathbf{x}\in V,\\ 0\,|\mathbf{x};t\rangle\,, & \mathbf{x}\notin V.\end{cases}.\end{aligned} \tag{2.41}$$

To obtain this result, we have used

$$\left[\hat{N}_V(t),\hat{\varphi}^\dagger(t,\mathbf{x})\right]|0\rangle = \begin{cases}\hat{\varphi}^\dagger(t,\mathbf{x}), & \mathbf{x}\in V,\\ 0, & \mathbf{x}\notin V.\end{cases} \tag{2.42}$$

Proof:

$$\begin{aligned}\left[\hat{N}_V(t),\hat{\varphi}^\dagger(t,\mathbf{x})\right] &= \int d^d\mathbf{y}\,\left[\hat{\varphi}^\dagger(t,\mathbf{y})\hat{\varphi}(t,\mathbf{y}),\hat{\varphi}^\dagger(t,\mathbf{x})\right] \qquad (2.43)\\ &= \int_V d^d\mathbf{y}\,\hat{\varphi}^\dagger(t,\mathbf{y})\delta^{(d)}(\mathbf{y}-\mathbf{x}) = \begin{cases}\hat{\varphi}^\dagger(t,\mathbf{x}), & \mathbf{x}\in V,\\ 0, & \mathbf{x}\notin V.\end{cases}\end{aligned}$$

Note that $\hat{N}_V(t)$ does not commute with H. Therefore, we emphasize that the above discussion refers to states at the very same time t.

We now consider multi-particle states. We first consider a two-particle state:

$$|\mathbf{x},\mathbf{y};t\rangle = \hat{\varphi}^\dagger(t,\mathbf{x})\hat{\varphi}^\dagger(t,\mathbf{y})\,|0\rangle = |\mathbf{y},\mathbf{x};t\rangle\,. \tag{2.44}$$

In the last equality, we used the quantization conditions (2.27). Therefore, we see that with such quantization conditions multiparticle states are bosonic.

We now consider how the operator $N_V(t)$ acts on these states. First note that

$$\hat{N}_V(t)\,|\mathbf{x},\mathbf{y};t\rangle = \left[\hat{N}_V(t),\hat{\varphi}^\dagger(t,\mathbf{x})\hat{\varphi}^\dagger(t,\mathbf{y})\right]|0\rangle\,. \tag{2.45}$$

and show that

Exercise for the reader

$$\hat{N}_V(t)\,|\mathbf{x},\mathbf{y};t\rangle = \begin{cases}2\,|\mathbf{x},\mathbf{y};t\rangle\,, & \text{if } \mathbf{x},\mathbf{y}\in V,\\ 1\,|\mathbf{x},\mathbf{y};t\rangle\,, & \text{if } \begin{cases}\mathbf{x}\in V;\mathbf{y}\notin V,\\ \mathbf{y}\in V;\mathbf{x}\notin V,\end{cases}\\ 0\,|\mathbf{x},\mathbf{y};t\rangle\,, & \text{if } \mathbf{x},\mathbf{y}\notin V.\end{cases} \tag{2.46}$$

Overall, we see that these results fit the interpretation of particle(s) located at $\mathbf{x}$ ($\mathbf{y}$).

2.1.6 Connection with Quantum Mechanics and the Probabilistic Interpretation

Since the fields are now operators (rather than wave functions, as in quantum mechanics), we would like to understand what has become of the probabilistic interpretation that we had in previous quantum mechanics courses.

One-particle Case:
The state $|\mathbf{x};t\rangle$ is an idealization. Real elements of the Hilbert space will have the following form

$$|\psi(t)\rangle = \int d^d\mathbf{x}\,\psi(\mathbf{x})\,|\mathbf{x};t\rangle\,, \quad \text{with} \quad \psi(\mathbf{x}) \in \mathcal{L}^2_{\mathbb{R}^d}. \tag{2.47}$$

We can also introduce the time dependence on the wave function:

$$|\psi(t)\rangle = \int d^d\mathbf{x}\,\psi(t,\mathbf{x})\,|\mathbf{x}\rangle\ , \tag{2.48}$$

where $|\mathbf{x}\rangle = |\mathbf{x};t=0\rangle$.

Irrespective of how we write the ket, it represents a real physical state and has a probabilistic interpretation, as we will see. If we compute the expectation value of N_V, we obtain

Probability density

$$\langle\psi(t)|\,\hat{N}_V(t)\,|\psi(t)\rangle = \int_V d^d\mathbf{x}\,|\psi(\mathbf{x})|^2 \quad \forall\, t. \tag{2.49}$$

Note that we can formally make V as small as we want. Then, $|\psi(\mathbf{x})|^2$ represents the probability density of finding a particle at position $\mathbf{x}$.

We now apply the Hamiltonian to this one-particle state

$$\begin{aligned} H\,|\psi(t)\rangle &= \int d^d\mathbf{x}, \psi(t,\mathbf{x})\hat{H}\hat{\varphi}^\dagger(\mathbf{x})\,|0\rangle \qquad (2.50)\\ &= \int d^d\mathbf{x}\ \psi(t,\mathbf{x}) \int d\mathbf{y}\ \hat{\varphi}^\dagger(\mathbf{y})\hat{h}_\mathbf{y}\hat{\varphi}(\mathbf{y})\hat{\varphi}^\dagger(\mathbf{x})\,|0\rangle\,. \end{aligned}$$

Using the commutation relations, the term $\hat{\varphi}^\dagger(\mathbf{x})\hat{\varphi}(\mathbf{y})$ applied to $|0\rangle$ is 0, and we obtain

$$\begin{aligned} H\,|\psi(t)\rangle &= \int d^d\mathbf{x}\ \psi(t,\mathbf{x}) \int d^d\mathbf{y}\ \hat{\varphi}^\dagger(\mathbf{y})\left(\hat{h}_\mathbf{y}\ \delta^{(d)}(\mathbf{x}-\mathbf{y})\right)|0\rangle \qquad (2.51)\\ &= \int d^d\mathbf{y}\ \hat{\varphi}^\dagger(\mathbf{y})\left(\hat{h}_\mathbf{y}\psi(t,\mathbf{y})\right)|0\rangle = \int d^d\mathbf{y}\left(\hat{h}_\mathbf{y}\psi(t,\mathbf{y})\right)\hat{\varphi}^\dagger(\mathbf{y})\,|0\rangle\,. \end{aligned}$$

On the other hand, we have

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \; . \tag{2.52}$$

Since this equality should hold for any $\hat{\varphi}$, we have

$$\boxed{\hat{h}_y\psi(t,\mathbf{y}) = i\partial_0\psi(t,\mathbf{y}),} \tag{2.53}$$

If we demand $H|\psi\rangle = E|\psi\rangle$, this can be achieved if

$$\boxed{\hat{h}_y\psi(\mathbf{y}) = E\psi(\mathbf{y}).} \tag{2.54}$$

These two equations, together with Eq. (2.49), nicely establish the connection with the Schrödinger equation and the probabilistic interpretation of the wave function introduced in earlier quantum-mechanics courses.

Two-Particle Case:
A general two-particle state reads as follows

$$|\psi(t)\rangle = \int d^d\mathbf{x}\int d^d\mathbf{y}\,\psi(t,\mathbf{x},\mathbf{y})\hat{\varphi}^\dagger(\mathbf{x})\hat{\varphi}^\dagger(\mathbf{y})\,|0\rangle \; . \tag{2.55}$$

As we know, see Eq. (2.44), the quantization conditions we are using imply that the particles obey Bose statistics. This has consequences for the wave function. Since it always appears multiplied by the fields (see Eq. (2.55)), only the symmetric part of the wave function is physically measurable and, without loss of generality, we can take

$$\psi(t,\mathbf{x},\mathbf{y}) = \psi(t,\mathbf{y},\mathbf{x}), \tag{2.56}$$

The probabilistic interpretation of the wave function also holds in this case:

$$\langle\psi(t)|N_V(t)|\psi(t)\rangle = 2\int_{V\times V} d^d\mathbf{x}d^d\mathbf{y}|\psi(\mathbf{x},\mathbf{y})|^2 + 1\int_{R-V}\dots \tag{2.57}$$

and something similar happens to the energy:

$$H|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle \quad \Longleftrightarrow \quad \boxed{(\hat{h}_x+\hat{h}_y)\psi(t;\mathbf{x},\mathbf{y}) = i\frac{\partial}{\partial t}\psi(t;\mathbf{x},\mathbf{y}).} \tag{2.58}$$

2.2 NR Fermions

Bose statistics is not the only option for quantization. We may also have Fermi statistics, which we analyze now.

The classical computation is equal than in Sec. 2.1.1 (we only change $a \to b$). Indeed, we obtain

$$H = \int d^d\mathbf{x}\, \psi^*(\mathbf{x})(\hat{h}_{\mathbf{x}}\psi(\mathbf{x})) = \sum_n e_n b_n^* b_n \tag{2.59}$$

for the Hamiltonian. Differences arise at the level of the quantization conditions, which we now take as follows

Anticommutation quantization condition

$$\{\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})\} \equiv \hat{\psi}(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y}) + \hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{x}) = \delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{2.60}$$
$$\{\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})\} = \{\hat{\psi}^\dagger(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})\} = 0.$$

These imply Fermi statistics:

$$\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y}) = -\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}^\dagger(\mathbf{x})\,, \tag{2.61}$$

and yield the following commutation relations for the b's:

Exercise

$$\{\hat{b}_n, \hat{b}_m^\dagger\} = \delta_{nm}, \quad \{\hat{b}_n, \hat{b}_m\} = \{\hat{b}_n^\dagger, \hat{b}_m^\dagger\} = 0. \tag{2.62}$$

As in the boson case, we may wonder whether the quantization condition in Eq. (2.60) or that in Eq. (2.62) is more fundamental.

The Hamiltonian operator is equal to the one in Eq. (2.59) transforming the b's into operators (simply by adding hats):

$$\hat{H} = \int d^d\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x})(\hat{h}_{\mathbf{x}}\hat{\psi}(\mathbf{x})) = \sum_n e_n \hat{b}_n^\dagger \hat{b}_n\,. \tag{2.63}$$

As in the boson case, this looks like an infinite number of decoupled oscillators with different frequencies. There will be important differences in the structure of the Fock space, though.

Fock Space Representation

The Fock space formalism still applies (oscillator interpretation). States can be written in the following way, as in the boson case:

$$|(N_1, N_2, \ldots, N_j, \ldots)\rangle \ . \tag{2.64}$$

Nevertheless, it's important to note that for fermions $\{\hat{\psi}, \hat{\psi}\} = 0$. This will severely constrain how the number operator(s)

$$\hat{N} = \sum_n e_n \hat{b}_n^\dagger \hat{b}_n, \quad \hat{N}_i = \hat{b}_i^\dagger \hat{b}_i \tag{2.65}$$

act on the kets, and, consequently, what states are allowed in nature.

Exercise

Prove that $\hat{N}_i$ is a projector (idempotent and hermitian) operator:

$$\hat{N}_i^\dagger = \hat{N}_i\,, \quad \textit{and} \quad \hat{N}_i^2 = \hat{N}_i. \tag{2.66}$$

Proof ($\hat{b}_i^\dagger \hat{b}_i^\dagger = 0$):

$$\hat{N}_i^2 = \hat{N}_i \cdot \hat{N}_i = \hat{b}_i^\dagger\, \hat{b}_i\, \hat{b}_i^\dagger\, \hat{b}_i = \hat{b}_i^\dagger\, \hat{b}_i - \hat{b}_i^\dagger \hat{b}_i^\dagger\, \hat{b}_i \hat{b}_i = \hat{N}_i \rightarrow \boxed{\hat{N}_i^2 = \hat{N}_i}. \tag{2.67}$$

From this result, we conclude that the values of N_i in Eq. (2.64) can only be 0 or 1. This is nothing but Pauli's exclusion principle: There cannot exist two particles with the same quantum numbers. Note that this is not a hypothesis (principle) in our case, but follows from the quantization conditions we have chosen.[3]

Time Evolution of Operators

The time evolution of fermions is similar to the case of bosons but there are some differences in the derivation. We first have

$$\hat{b}_n^\dagger(t) = e^{it\hat{H}} \hat{b}_n^\dagger e^{-it\hat{H}}, \tag{2.68}$$

which can be written as a Hamilton's equation

$$\frac{d}{dt} \hat{b}_n^\dagger(t) = -i[\hat{b}_n^\dagger(t), \hat{H}]\,. \tag{2.69}$$

The solution to this equation is given in the following exercise

[3]In relativistic four dimensional quantum theories Fermi and Bose statistics are the only possibilities for quantization conditions. The situation is different in three dimensions, where parastatistics (anyons) may appear (see, for instance, the discussion in [4]).

Exercise

Show that[a]

$$\frac{d}{dt}\hat{b}_n^\dagger(t) = -i[\hat{b}_n^\dagger(t), \hat{H}] = -i\left[e^{it\hat{H}}\hat{b}_n^\dagger e^{-it\hat{H}}, \sum_m e_m \hat{b}_m^\dagger \hat{b}_m\right] = ie_n\hat{b}_n^\dagger(t)$$
$$\Rightarrow \hat{b}_n^\dagger(t) = \hat{b}_n^\dagger e^{-ie_nt}, \quad \hat{b}_n(t) = \hat{b}_n e^{-ie_nt}. \tag{2.70}$$

[a] Here we apply the following identity $[AB, C] = A\{B, C\} - \{A, C\}B$ instead of $[AB, C] = A[B, C] + [A, C]B$, which was used for bosons. It should be noted that we do not need more for fermions, but for bosons we can apply the other identity as many times as necessary.

Localizability (and particle interpretation)

The localizability properties are equivalent to the bosonic case. We can define a number operator in a finite volume V:

$$\hat{N}_V(t) = \int_V d^d\mathbf{x}\, \hat{\varphi}^\dagger(t, \mathbf{x})\hat{\varphi}(t, \mathbf{x}). \tag{2.71}$$

Using that (where in the second equality we have applied, again, the identity $[AB, C] = A\{B, C\} - \{A, C\}B$)

$$[\hat{N}_V(t), \hat{\varphi}^\dagger(t, \mathbf{y})] = \int_V d^d\mathbf{x}\, [\hat{\varphi}^\dagger(t, \mathbf{x})\hat{\varphi}(t, \mathbf{x}), \hat{\varphi}^\dagger(t, \mathbf{y})]$$
$$= \int_V d^d\mathbf{x}\, \hat{\varphi}^\dagger(t, \mathbf{x}) \left\{\hat{\varphi}(t, \mathbf{x}), \hat{\varphi}^\dagger(t, \mathbf{y})\right\}$$
$$= \int_V d^d\mathbf{x}\, \hat{\varphi}^\dagger(t, \mathbf{x})\delta^{(d)}(\mathbf{x} - \mathbf{y}) = \begin{cases} \hat{\varphi}^\dagger(t, \mathbf{x}), & \mathbf{x} \in V \\ 0, & \mathbf{x} \notin V \end{cases}, \tag{2.72}$$

we obtain

One-particle Case

$$\hat{N}_V(t)\hat{\varphi}^\dagger(t, \mathbf{x}) |0\rangle = \begin{cases} 1\, |\mathbf{x}; t\rangle\,, & \mathbf{x} \in V \\ 0\, |\mathbf{x}; t\rangle\,, & \mathbf{x} \notin V \end{cases}. \tag{2.73}$$

An analogous result can be obtained for the two-particle state (see the discussion in the bosonic case). This is left as an exercise for the reader.

Note that the starting point in Eqs. (2.69) and (2.72) involves commutators. Therefore, the initial expressions are equal for bosons and fermions. The difference arises when we express the operators ($\hat{H}$, $\hat{N}$, $\hat{N}_V$) in terms of the fields. At that stage, we are compelled to use different identities for each case in order to take advantage of the specific commutation or anticommutation relations that hold due to the quantization conditions given in Eq. (2.13) and Eq. (2.61).

Connection with quantum mechanics and probabilistic interpretation

The connection with NR quantum mechanics and the probabilistic interpretation follows an analogous discussion to the boson case. If we have the one-particle state

$$|\psi(t)\rangle = \int d^d\mathbf{x}\,\psi(\mathbf{x})\,|\mathbf{x};t\rangle\,, \quad \text{with} \quad \psi(\mathbf{x}) \in \mathcal{L}^2_{\mathbb{R}^d}, \tag{2.74}$$

we obtain

Exercise

$$\langle\psi(t)|\,\hat{N}_V(t)\,|\psi(t)\rangle = \int_V d^d\mathbf{x}\,|\psi(\mathbf{x})|^2. \tag{2.75}$$

Similarly,

$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle \to \hat{h}_{\mathbf{x}}\psi(\mathbf{x}) = E\psi(\mathbf{x}). \tag{2.76}$$

Exercise

Perform the same analysis for the two-particle case.

2.3 Exercises

1. Consider the following Lagrangian

$$L = \int d^d\mathbf{x}\varphi^\dagger(\mathbf{x})\left\{i\partial^0 + \frac{\boldsymbol{\nabla}^2}{2m_1}\right\}\varphi(\mathbf{x}) + \int d^d\mathbf{x}\chi_c^\dagger(\mathbf{x})\left\{i\partial^0 + \frac{\boldsymbol{\nabla}^2}{2m_2}\right\}\chi_c(\mathbf{x})$$
$$- \int d^d\mathbf{x}_1 d^d\mathbf{x}_2\varphi^\dagger(t,\mathbf{x}_1)\chi_c^\dagger(t,\mathbf{x}_2)V(\mathbf{x}_1-\mathbf{x}_2)\chi_c(t,\mathbf{x}_2)\varphi(t,\mathbf{x}_1)\,, \tag{2.77}$$

where φ is a fermion and χ_c an anti-fermion. Obtain H, project to the following one fermion-antifermion state:

$$|\psi\rangle = \int d^d\mathbf{x}_1\,d^d\mathbf{x}_2\,\psi(\mathbf{x}_1,\mathbf{x}_2)\,\hat{\varphi}^\dagger(\mathbf{x}_1)\hat{\chi}_c^\dagger(\mathbf{x}_2)\,|0\rangle\,, \tag{2.78}$$

and show that

$$\hat{H}\,|\psi(t)\rangle = i\frac{\partial}{\partial t}\,|\psi(t)\rangle$$
$$\Updownarrow$$
$$\left[-\frac{\boldsymbol{\nabla}^2_{x_1}}{2m_1} - \frac{\boldsymbol{\nabla}^2_{x_2}}{2m_2} + V(|\mathbf{x}_1 - \mathbf{x}_2|)\right]\psi(t;\mathbf{x}_1,\mathbf{x}_2) = i\frac{\partial}{\partial t}\psi(t;\mathbf{x}_1,\mathbf{x}_2) \tag{2.79}$$

2. **Number operator** for free NR particles
We define the number operator as

$$\hat{N} = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}$$

a) Show that

$$\hat{N}\hat{a}^\dagger_{\mathbf{k}}|0\rangle = 1\hat{a}^\dagger_{\mathbf{k}}|0\rangle \qquad \hat{N}\hat{a}^\dagger_{\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}}|0\rangle = 2\hat{a}^\dagger_{\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}}|0\rangle \cdots$$

b) Show that

$$\hat{N} = \int d^d\mathbf{x}\hat{\varphi}^\dagger(t,\mathbf{x})\hat{\varphi}(t,\mathbf{x})$$

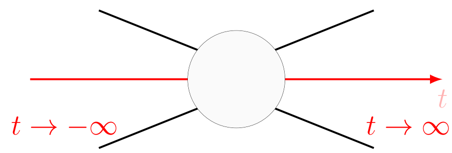
c) If we define

$$\hat{N}_V(t) = \int_V d^d\mathbf{x}\hat{\varphi}^\dagger(t,\mathbf{x})\hat{\varphi}(t,\mathbf{x})$$

compute $[\hat{N}_V(t),\hat{\varphi}(t,\mathbf{x})]$ and $[\hat{N}_V(t),\hat{N}_{V'}(t)]$. Interpret these results.

3 Fock Space and Asymptotic States

[1] Scattering processes of relativistic particles can be represented as follows



These processes are described in terms of asymptotic states. Asymptotic states correspond to free particles in the infinite past and in the infinite future: They are too far apart to interact with each other (or we will assume that such interaction can be neglected[2]), but they may eventually interact with an external detector (e.g., via electromagnetic or gravitational interactions). Therefore, their interaction occurs, at most, only for a finite period of time. Due to relativity, this interaction also occurs within a finite spatial volume. Thus, the interaction between these states takes place within a finite space-time volume. This is one of the reasons why constructing the Fock space of free particles is so important in QFTs. The concept of Fock space is essential for describing these multiparticle states that arise in scattering processes.

Therefore, our first task is to characterize these asymptotic states.

Asymptotic States and Free Particles

- Free particles are one-particle unitary irreducible representations of the proper orthochronous inhomogeneous Lorentz group (Poincaré group for short).
- Parity and time-reversal transformations are not included in the group. This is the reason we do not consider them in the characterization of free particles we discuss in this chapter.
- The Hilbert space that will describe these scattering processes is the direct product of these one-particle unitary irreducible representations of the proper orthochronous inhomogeneous Lorentz group. We refer to this object as Fock space.

[1]In this chapter, we use capital letters to denote operators, except when referring to creation and annihilation operators, for which we keep using the $\hat{}$ notation.

[2]There are qualifications to this statement, which we do not address in this book.

To fully characterize the Hilbert space associated with one-particle unitary irreducible representations of the Poincaré group, we must first understand how symmetries are realized in quantum theories, and then apply this understanding to the specific Lie group at hand (Poincaré). We discuss this issue in the following section.

3.1 Connection Between Symmetry Groups and Physics

We consider that we have a connected Lie group. We look for unitary representations that can be obtained from its associated Lie algebra. Given a group G, with associated Lie algebra L, the connection goes as follows:

$$\boxed{G \longrightarrow L \longrightarrow T_N(\tilde{A}) \longrightarrow T_N(g) \equiv e^{T_N(\tilde{A})} \longrightarrow \mathcal{H}_N,} \tag{3.1}$$

where $\tilde{A} \in L$. $T_N(\tilde{A})$ is an anti-Hermitian representation of $\tilde{A}$ (and similarly for all elements of L): so that $e^{T_N(\tilde{A})} \equiv U_N(g)$ are unitary operators. In general, N will be a collection of numbers that completely characterizes the Hilbert space to which the unitary representation of G applies[3]. Overall,

- If $T_N(\tilde{A})^\dagger = -T_N(\tilde{A})$, then $T_N(g)$ is unitary.
- The Hilbert space $\mathcal{H}_N$ carries the representation and is the space where physical states live.
- Applying $U_N(g)$ $\forall g$ allows us to move between all elements of $\mathcal{H}_N$.

3.2 Example: SO(3)

In this section, we apply Eq. (3.1) to the group $G = SO(3)$. The unitary irreducible representations of this symmetry group act on Hilbert spaces of different dimensions depending on the physical system under consideration. Below, we present a few key examples of representations of the $L = so(3)$ Lie algebra (which is part of the Lorentz algebra), relevant to the Poincaré group.

We have that $\tilde{A} = -i \sum \alpha_i \mathbf{M}^i$, and $\mathbf{M}$ are the 3×3 sub-matrices in Eq. (1.81). $T_N(\tilde{A})$ is an anti-Hermitian (unitary) representation of the $so(3)$ Lie algebra in a Hilbert space (and N is the dimension of the V-space-$\mathbb{C}$).

[3] As we will see later, for SO(3), one can take N to be the dimension of the V-space-$\mathbb{C}$, i.e of the Hilbert space on which the operators act.

Unitary Representation

The unitary operator implementing the rotation is

$$\hat{U}_N(R) = e^{-i\theta\,\hat{n}\cdot T_N(\mathbf{M})}, \tag{3.2}$$

where $\hat{n}$ is the rotation axis, θ the angle, and $T_N(\mathbf{M}^k)$ are Hermitian operators, which we name as

$$\mathbf{J}^i \equiv T_N(\mathbf{M}^i) \qquad \Rightarrow \qquad [\mathbf{J}^i, \mathbf{J}^j] = i\epsilon^{ijk}\mathbf{J}^k. \tag{3.3}$$

The associated Hilbert space $\mathcal{H}_N$ is characterized by the eigenstates of the Casimir operator $\mathbf{J}^2$, and spanned by the eigenvectors of $\mathbf{J}^3$ (or any other linear combination of $\mathbf{J}$'s)

$$\mathcal{H}_N = |\text{casimir, (basis)}\rangle = \text{span}\left\{\left|J^2, J_z\right\rangle\right\}. \tag{3.4}$$

The representation $U_N(R)$ acts on the elements of this Hilbert space

$$|\psi\rangle \in \mathcal{H}. \tag{3.5}$$

Key Idea

Hilbert space carries a unitary irreducible representation of the symmetry group.

Examples:

- **Spin-$\frac{1}{2}$ representation (dimension 2)**

$$T_2(M^i) = \frac{\sigma^i}{2}, \tag{3.6}$$

  where σ^i are the Pauli matrices. This is the representation used for spin-$\frac{1}{2}$ particles such as electrons. The corresponding Hilbert space has dimension 2.

- **Spin-1 representation (dimension 3)**

$$T_3(M^i) = M^i, \tag{3.7}$$

  where M^i are the generators of rotations acting on three dimensional vectors. This representation is relevant for massive vector bosons. Its Hilbert space is 3-dimensional.

- **Infinite-dimensional representation (wavefunction in position space)**

$$T_\infty(M_z) \equiv L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{3.8}$$

  where the generators act as differential operators on wavefunctions $\psi(\mathbf{x}) \in \mathcal{L}^2_{\mathbb{R}^3}$. This representation is infinite-dimensional and corresponds to the standard angular momentum operator in quantum mechanics.

These examples illustrate how different physical systems correspond to different representations of the same symmetry Lie algebra.

Particles are associated with Hilbert spaces that carry a unitary irreducible representation of the Poincaré group. We now discuss how to characterize these Hilbert spaces. This is achieved using the Wigner method.

3.3 The Wigner Method*

Aim. The goal is to obtain the **unitary irreducible representations of the Poincaré Lie algebra for *one-particle states***.

The Fock space that describes physical processes is then constructed as an infinite direct product of these irreducible representations—but not all of them—since not all irreducible representations are realized in nature as elementary particles (we neither observe all masses nor all spins).

Important note: Whether or not *parity* is considered a symmetry makes a difference in the construction of these irreducible representations. This is especially relevant in the massless case. However, one must bear in mind that parity is *not* a fundamental symmetry of nature, and it is not included in $SO_o(1,3)$.

3.3.1 One-Particle States

First, we need to identify the quantities that remain invariant under Poincaré transformations. These are the Casimir operators of the Poincaré group. We will obtain them below. They yield quantum numbers that are equal for all elements of the Hilbert space describing one-particle states. Next, we look for operators that commute with $H = P^0$.

- We assume $P^0 > 0$ (positive energy condition).
- We take P^μ as the initial input ($[P^\mu, P^\nu] = 0$) to classify the states and look for other quantities that commute with P^0.

Definition of One-Particle States

$$|1 \text{ particle state}\rangle = \left|p^\mu, .\overset{?}{.}.\right\rangle \quad \text{(labels still to be determined)}. \tag{3.9}$$

The one-particle state will be characterized by its momentum p^μ and other quantum numbers to be identified.

Classification of states from p^2 and p^0

We take the first Casimir invariant of the Poincaré Lie algebra to be

$$C_1 = P^\mu P_\mu \xrightarrow{\textit{eigenvalues}} p^2 = m^2. \tag{3.10}$$

The momentum operator satisfies (we omit the possible action on other quantum numbers in this discussion)

$$P^\mu |p\rangle = p^\mu |p\rangle . \tag{3.11}$$

We now focus on how p^μ transforms under Lorentz transformations

$$U(\Lambda, a) |p\rangle \propto |\Lambda p\rangle , \qquad P^\mu |\Lambda p\rangle = (\Lambda p)^\mu |\Lambda p\rangle .$$

Since

$$(\Lambda p)^2 = (\Lambda p)^\mu (\Lambda p)_\mu = p^2,$$

we conclude that p^2 is Lorentz invariant. On top of that we have that

Important Observation

The sign of p^0 does not change under proper orthochronous Lorentz transformations.[a]

[a] See also pp. 37,38 in [6].

Thus, one-particle states are organized according to the possible values p^2 and p^0:

(i) $p^2 = m^2 > 0, \quad p^0 = \sqrt{m^2 + \mathbf{p}^2} \equiv E_p > 0$: Physical (massive modes).

(ii) $p^2 = 0, \quad p^0 = |\mathbf{p}| \equiv E_p > 0$: Physical (massless modes).

(iii) $p^\mu = 0$: Physical (vacuum).

(iv) $p^2 < 0$: Non-physical (Note: Lorentz transformations can flip the sign of p^0 if p^μ is space-like).

(v) $p^2 = m^2 > 0, \quad p^0 < 0$: Non-physical.

(vi) $p^2 = 0, \quad p^0 < 0$: Non-physical.

3.3.2 Little Group

To further characterize one-particle states, we introduce the concept of the little group of p^μ.

Definition

The little group $\mathcal{W}(p)$ is the subgroup of the Lorentz group that leaves p^μ invariant.

Little Group: Vacuum
The little group of $p^\mu = 0$, the vacuum, is the full Lorentz group.

3.3.3 Little Group: Massive Case

To determine the little group of massive modes ($p^2 > 0$, $p^0 > 0$), we use the following Theorem.

Theorem 1 (Page 39 [6])

Given a four-vector v^μ, its little group depends solely on the sign of v^2. In other words, if $\text{sign}(v^2) = \text{sign}(v'^2)$, then

$$\mathcal{W}(v) \cong \mathcal{W}(v'). \tag{3.12}$$

That is, the little groups of v and v' are isomorphic.

This theorem applies to time-like, space-like, and light-like vectors. For time-like vectors, we give below an explicit demonstration.

Strategy: Choose a reference vector p^μ such that the little group is easy to identify. This way, we can fully determine the one-particle unitary irreducible representation for massive modes. The simplest choice is to take the rest frame:

$$k^\mu = (m, 0, 0, 0).$$

In this frame, the little group is the **rotation group**:

$$\mathcal{W}(k) = SO(3)\,,$$

and we know all the irreducible representations of $SO(3)$!

Any other time-like four-vector can be obtained via a Lorentz transformation from k^μ. Thus, using the previous theorem

$$\mathcal{W}(p^\mu) \cong SO(3),$$

for all time-like p^μ vectors.

Explicit Demonstration for Massive Modes

We now explicitly show that $\mathcal{W}(p^\mu) \cong SO(3)$ for arbitrary time-like vectors. An arbitrary time-like vector can always be written in the following way:

$$p^\mu = [L(p)]^\mu_{\ \nu} k^\nu, \qquad k^\mu = (m, \mathbf{0}),$$

where k^ν is the rest frame 4-momentum, and $L(p)$ is a Lorentz transformation that can be taken to be a boost in the p direction.[4]

We then define the state $|k, \sigma\rangle$. σ was what we named $\mathbf{J}^3$ in Eq. (3.4), but we could have chosen a different direction. It spans the basis of the Hilbert space for a given $\mathbf{J}^2$. From this state, we can define arbitrary states at different momenta by applying unitary Lorentz transformations:

$$|p, \sigma\rangle \equiv U(L(p))\, |k, \sigma\rangle\ . \tag{3.13}$$

Both are eigenstates of the momentum operator:

$$P^\mu\, |p, \sigma\rangle = p^\mu\, |p, \sigma\rangle\ , \quad P^\mu\, |k, \sigma\rangle = k^\mu\, |k, \sigma\rangle\ .$$

Now consider another time-like 4-momentum $p'^\mu = \Lambda^\mu_{\ \nu} p^\nu$. Then,

$$U(\Lambda)\, |p, \sigma\rangle$$

will be an eigenstate of P^μ with eigenvalue p'^μ. Moreover, using the group structure of the unitary representation, we have

$$\begin{aligned} U(\Lambda)\, |p, \sigma\rangle &= U(\Lambda) U(L(p))\, |k, \sigma\rangle = U(L(\Lambda p)) \left[U^{-1}(L(\Lambda p)) U(\Lambda) U(L(p)) \right] |k, \sigma\rangle \\ &= U(L(\Lambda p)) \left[U(L^{-1}(\Lambda p)) U(\Lambda) U(L(p)) \right] |k, \sigma\rangle \\ &= U(L(\Lambda p)) U(L^{-1}(\Lambda p) \Lambda L(p))\, |k, \sigma\rangle\ . \end{aligned} \tag{3.14}$$

Observation: The operator $L^{-1}(\Lambda p)\Lambda L(p)$ leaves k^μ invariant,

$$\left(L^{-1}(\Lambda p) \Lambda L(p)\, k \right)^\mu = k^\mu,$$

where $(\Lambda L(p) k)^\mu \equiv p'^\mu$, $(L(p)k)^\mu \equiv p^\mu$. Therefore, this corresponds to a rotation R in the rest frame. That is,

$$\mathcal{W}(k^\mu) \ni L^{-1}(\Lambda p) \Lambda L(p) = e^{-i \boldsymbol{\Theta} \cdot \mathbf{M}} = R.$$

[4] $L(p)$ is an element of $SO_o(1,3)$, but we denote it this way (instead of Λ) to emphasize that it is characterized by the momentum p.

Hence, we can write the transformation law as follows

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= U\left(L(\Lambda p)\right)\sum_{\sigma'} T_{\sigma'\sigma}(R)\,|k,\sigma'\rangle = \sum_{\sigma'} T_{\sigma'\sigma}(R) U\left(L(\Lambda p)\right)|k,\sigma'\rangle \\ &= \sum_{\sigma'} T_{\sigma'\sigma}(R)\,|\Lambda p,\sigma'\rangle\,, \end{aligned} \tag{3.15}$$

where $T_{\sigma'\sigma}(R)$ is a unitary representation of the rotation group $SO(3)$.

Result

The Wigner little group $\mathcal{W}(p^\mu)$ for time-like momenta is isomorphic to $SO(3)$, and the states transform under unitary representations of this rotation group.

3.3.4 Second Casimir Operator and Spin

The second Casimir invariant of the Poincaré group, C_2, is related to spin in the massive case, and it must commute with all generators of the Poincaré group. It cannot simply be $\mathbf{J}^2$, since $[\mathbf{J}^2, \mathbf{K}^i] \neq 0$. We define it in the following way:

$$C_2 \equiv W^2 = W^\mu W_\mu, \tag{3.16}$$

where W^μ is the Pauli–Lubanski pseudovector (see Eq. (1.94)):

$$\boxed{W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma\,.}$$

By construction, it satisfies

$$W_\mu P^\mu = 0.$$

If we apply W_0 to a particle at rest with $k^\mu = (m, \mathbf{0})$, we get

$$W_0\,|k,\sigma\rangle = 0.$$

Therefore, W^μ is a space-like vector. Explicitly, in the rest frame, we have

$$W_i = -\frac{1}{2}\epsilon_{i\nu\rho\sigma} J^{\nu\rho} P^\sigma \doteq -\frac{m}{2}\epsilon_{ijk0} J^{jk} = -m\mathbf{J}^i.$$

Conclusion: When applied to a particle at rest,

$$W^2 \doteq -m^2 \mathbf{J}^2.$$

Since W^2 defines a Casimir operator—that is, it commutes with all generators of the Poincaré group—we can compute its eigenvalues in the rest frame, and they will remain unchanged in any other frame:

$$C_2 = W_\mu W^\mu \quad \xrightarrow{\textit{eigenvalues}} \quad w^2 = -m^2 j(j+1), \quad \text{with } j \text{ being the spin.}$$

Casimir Operators of the Poincaré Group

The Poincaré group is a rank-2 Lie group. Therefore, it has only two independent Casimir invariants

$$C_1 = P^\mu P_\mu, \qquad C_2 = W^\mu W_\mu.$$

A massive particle state can now be completely characterized by the following quantum numbers:

$$\left|(m^2, j); \mathbf{p}, \sigma\right\rangle,$$

with m^2 and j the eigenvalues of the Casimirs, and $\mathbf{p}, \sigma$ labeling the momentum and spin component.

3.3.5 Light-like Vectors and their Little Group

We now consider light-like vectors. They satisfy

$$p^2 = g_{\mu\nu} p^\mu p^\nu = 0, \qquad p^0 > 0.$$

To determine its little group, we again use the property (3.12) obtained in Theorem 1. We choose a standard momentum

$$k^\mu = (k, 0, 0, k), \quad k > 0\,, \tag{3.17}$$

and perform a proper orthochronous Lorentz transformation that leaves k^μ invariant, i.e. we apply an element of $\mathrm{SO}_o(1,3)$ such that

$$\left(e^{-i\mathbf{\Phi}\cdot\mathbf{K}} e^{-i\mathbf{\Theta}\cdot\mathbf{M}}\right)^\mu_{\ \nu} k^\nu = k^\mu.$$

We perform the analysis infinitesimally. The allowed transformations are parametrized in terms of 3 real numbers:

$$\Lambda = I - i\Theta_3 M_z - iu(K_x - M_y) - iv(K_y + M_x).$$

These combinations define the generators

$$N_1 \equiv K_x - M_y, \qquad N_2 \equiv K_y + M_x.$$

Together with M_z, these generators satisfy the Lie algebra of $E(2)$:

$$[M_z, N_1] = iN_2, \qquad [M_z, N_2] = -iN_1, \qquad [N_1, N_2] = 0.$$

$E(2)$ is the two-dimensional Euclidean group (translations and rotations in a two-dimensional Euclidean manifold), also known as $ISO(2)$.

Using the property (3.12) obtained in Theorem 1, we have

Little Group for Light-like Vectors

The little group of a light-like vector is isomorphic to the Euclidean group in two dimensions

$$\mathcal{W}(P^\mu) \cong ISO(2).$$

Unitary representations of the Little Group $E(2)$ and Physical Implications

The group structure of $E(2)$ and its irreducible representations are known, since $E(2)$ corresponds to a two dimensional Euclidean manifold. Therefore, we only have translations and rotations in the plane. We define

$$P^2 \Rightarrow N^2 \equiv N_1^2 + N_2^2 \,, \tag{3.18}$$

where (N_1, N_2) plays a role analogous to momentum in Euclidean space (but it is not!).

We look for unitary irreducible representations of these generators. The eigenvalues of (the unitary representations of) N^2 are denoted by

$$n^2 = n_1^2 + n_2^2 \,. \tag{3.19}$$

It is interesting to see how the operator N^2 relates to W^2. If we apply W^2 to a state characterized by k^μ defined in Eq. (3.17), we obtain

$$w^2 = -w_k^2 n^2 \,. \tag{3.20}$$

Observation. Only the eigenvalue $n^2 = 0$ ($\Rightarrow n_1 = n_2 = 0$) appears in nature. Therefore, the only eigenvalue of W^2 that is realized in nature is zero:

$$w^2 = 0.$$

Wigner's original paper provided some arguments as to why $w^2 \neq 0$ representations cannot appear in physical systems. Other references, such as [4, 6], simply state that such states *do not show up in nature*. Nevertheless, attempts to construct nontrivial interacting theories with $n^2 \neq 0$ occasionally appear in the scientific literature.

To fully characterize the state, we are left with rotations in this two-dimensional Euclidean plane:

$$SO(2) \cong U(1).$$

The representations of this group are one dimensional. We will label them σ, in analogy to spin in the massive case, and name them helicity. It is the projection of spin onto the momentum direction of the massless particle. The value of the helicity, a priori, could be any real number, and not restricted to integer or semi-integer values. We need to consider the full ISO(2) group and use topological arguments based on the fact that the Lorentz group is not simply connected (see [4]) to restrict the possible values of helicity to be integers or semi-integers (as the spin of massive particles). Therefore, most of the time, we will use the same terminology for both massive and massless particles. However, it is important to note that, even if we call them both "spin", in general we cannot go continuously from the massive to the massless case because the dimension N of the associated Hilbert space is different.

Extra remarks on helicity

- Helicity of massless particles is Lorentz invariant.
- States with opposite helicities (same magnitude, different sign) do not need to be related a priori. They would be related only if we enlarge the symmetry group to include parity.

Finally, note that W^μ and P^μ are not independent in the massless case if $N^2 = 0$. In this case, we have $P^2 = 0$, $W^2 = 0$ and $W \cdot P = 0$. Therefore, they are proportional

$$W^\mu \propto P^\mu \xrightarrow{\textit{implies}} \qquad (W^\mu - \lambda P^\mu)\,|p\rangle = 0\,,$$

and it can be seen that $\lambda = \sigma$, the helicity of the particle.

3.4 Summary. Fock Space and Operator Formalism

One-particle unitary irreducible representations of the Poincaré Group

In summary, from the classification of the unitary irreducible representations obtained using Wigner's method, a one-particle state can be characterized by the following quantum numbers

$$\left|(m^2, w^2); \mathbf{p}, \sigma, n\right\rangle, \tag{3.21}$$

where m^2 denotes the squared mass, w^2 is the W^2 operator eigenvalue, $\mathbf{p}$ is the three-momentum, σ corresponds to the third component of the spin or to the helicity (for massive particles we can also work with the helicity), and n denotes possible additional quantum numbers. For massive particles, $w^2 = -m^2 j(j+1)$, and it can be traded for j, the spin of the particles. For massless particles, both $p^2 = m^2$ and w^2 are equal to zero. From the context of the computation, it will be clear whether we are dealing with a massive or massless particle, as well as what the spin of the particle is. Therefore, we will often use the following shorthand notation:

$$|\mathbf{p}, \sigma, n\rangle . \tag{3.22}$$

We now address the construction of the Fock space from these one-particle states. The creation/annihilation formalism is the most convenient.

Creation and Annihilation Operators

We build the Fock space (i.e., the full Hilbert space) using the formalism of creation and annihilation operators:

$$|\mathbf{p}, \sigma\rangle \equiv \hat{a}^\dagger_{\mathbf{p},\sigma} |0\rangle , \tag{3.23}$$

where

$$|0\rangle \equiv \text{Fock vacuum} = \text{state of minimal energy}, \tag{3.24}$$

is the vacuum state. For two-particle states, we apply two creation operators:

$$|\mathbf{p_1}, \sigma_\mathbf{1}; \mathbf{p_2}, \sigma_\mathbf{2}\rangle \equiv \hat{a}^\dagger_{\mathbf{p_1},\sigma_\mathbf{1}} \hat{a}^\dagger_{\mathbf{p_2},\sigma_\mathbf{2}} |0\rangle . \tag{3.25}$$

This procedure extends to higher numbers of particles by applying the necessary amount of creation operators.

Commutation and Anticommutation Relations

As we have already discussed in the NR case in Chapter 2, the statistics is fixed by the quantization condition:

$$\begin{array}{cc} \underline{\text{For bosons}} & \underline{\text{For fermions}} \\ \left[\hat{a}_{\mathbf{p},\sigma}, \hat{a}^\dagger_{\mathbf{p}',\sigma'}\right] = (2\pi)^d \delta^{(d)}(\mathbf{p} - \mathbf{p}')\delta_{\sigma,\sigma'}, & \left\{\hat{a}_{\mathbf{p},\sigma}, \hat{a}^\dagger_{\mathbf{p}',\sigma'}\right\} = (2\pi)^d \delta^{(d)}(\mathbf{p} - \mathbf{p}')\delta_{\sigma,\sigma'} , \\ \left[\hat{a}_{\mathbf{p},\sigma}, \hat{a}_{\mathbf{p}',\sigma'}\right] = \left[\hat{a}^\dagger_{\mathbf{p},\sigma}, \hat{a}^\dagger_{\mathbf{p}',\sigma'}\right] = 0. & \left\{\hat{a}_{\mathbf{p},\sigma}, \hat{a}_{\mathbf{p}',\sigma'}\right\} = \left\{\hat{a}^\dagger_{\mathbf{p},\sigma}, \hat{a}^\dagger_{\mathbf{p}',\sigma'}\right\} = 0 , \end{array} \tag{3.26}$$

where we do not write the Casimirs, which we take to be equal (otherwise the commutator is 0).

Fractional statistics (e.g., anyons) could theoretically exist, but they do not occur in (3+1)-dimensional space-time (see [4]). Also, at this stage in the construction of the Fock space, statistics cannot yet be linked to spin. We can create a Fock space made of fermions or bosons regardless of their spin. The link only appears in interacting relativistic field theories: Integer-spin particles need to fulfil Bose statistics, and half-integer Fermi statistics. We do not prove this, though, in this book. For the proof, see the discussion in [4].

Lorentz Invariance and Boosts

The above commutation relations are not explicitly Lorentz invariant. To illustrate this, consider a boost in the third spatial direction[5]

$$p'_3 = \gamma(p_3 + \beta E_p), \quad E_{p'} = \gamma(E_p + \beta p_3). \tag{3.27}$$

Applying these transformations to the delta function, we obtain

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0) \Rightarrow \delta^{(d)}(\mathbf{p} - \mathbf{q}) = \delta^{(d)}(\mathbf{p}' - \mathbf{q}')\frac{E_{p'}}{E_p}. \tag{3.28}$$

This result highlights the non-trivial behavior of the commutators under Lorentz transformations. Therefore, we choose to normalize the creation and annihilation operators in the following way:

Normalization Conditions

$$\underline{\text{Bosons}} \;\rightarrow\; \left[\hat{a}_{\mathbf{p},\sigma}, \hat{a}^\dagger_{\mathbf{p}',\sigma'}\right] = 2E_p\, \delta^{(d)}(\mathbf{p} - \mathbf{p}')\, \delta_{\sigma\sigma'}. \tag{3.29}$$

$$\underline{\text{Fermions}} \;\rightarrow\; \left\{\hat{a}_{\mathbf{p},\sigma}, \hat{a}^\dagger_{\mathbf{p}',\sigma'}\right\} = \frac{2E_p}{2m}\, \delta^{(d)}(\mathbf{p} - \mathbf{p}')\, \delta_{\sigma\sigma'}. \tag{3.30}$$

For fermions, we have followed the popular choice (see [7]) of dividing the commutation relation by $2m$. This convention allows us to smoothly connect with the normalization used for the commutation relations in NR quantum mechanics (where $E/m \simeq 1$).

Pauli Exclusion Principle from Anticommutation

We remind the reader that, in the case of fermions, we cannot have two-particle states with the same quantum numbers, since

$$|\mathbf{p}, \sigma; \mathbf{p}, \sigma\rangle = -\,|\mathbf{p}, \sigma; \mathbf{p}, \sigma\rangle \Rightarrow |\mathbf{p}, \sigma; \mathbf{p}, \sigma\rangle = 0. \tag{3.31}$$

As we already discussed in Chapter 2, this is the Pauli Exclusion Principle. We stress again that this is not a principle we impose—it follows directly from the anticommutation relations.

Operators

We slightly change notation in the following, writing: $\hat{a}_{\mathbf{p},\sigma} \rightarrow \hat{a}_\sigma(\mathbf{p})$. We can easily write the operators $\hat{P}^\mu$ and $\hat{N}$ in terms of these creation and annihilation operators.

[5] We follow the discussion in p. 23 of [10].

$$\hat{P}^0 = \hat{H}^{(0)} = \int d\tilde{k} \sum_{\sigma} \omega_k \, \hat{a}_\sigma^\dagger(\mathbf{k})\hat{a}_\sigma(\mathbf{k}), \qquad \hat{\mathbf{P}} = \int d\tilde{k} \sum_{\sigma} \mathbf{k} \, \hat{a}_\sigma^\dagger(\mathbf{k})\hat{a}_\sigma(\mathbf{k}). \quad (3.32)$$

$$\hat{N} = \int d\tilde{k} \sum_{\sigma} \hat{a}_\sigma^\dagger(\mathbf{k})\hat{a}_\sigma(\mathbf{k}). \quad (3.33)$$

Notation $(d = 3)$

$$d\tilde{k}\Big|_{\text{boson}} \equiv \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{2\omega_k}, \qquad d\tilde{k}\Big|_{\text{fermion}} \equiv \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{m}{\omega_k}, \qquad \omega_k \equiv E_k = \sqrt{m^2 + \mathbf{k}^2}. \quad (3.34)$$

Note that this phase space integral can be written in a Lorentz invariant way:

Exercise

Show that

$$\int \frac{d^D p}{(2\pi)^D}(2\pi)\delta(p^2 - m^2)\theta(p^0) = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{2E_p}\,.$$

Once we realize that one can write the phase factor in an explicitly Lorentz invariant way, we may consider rewriting it in alternative ways. One example is the following:

Exercise

Determine whether there is any value of Z that fulfils this equality ($m > 0$):

$$\int \frac{d^D p}{(2\pi)^D}(2\pi)\delta(p^2 - m^2)\theta(p^0) = Z \int \frac{dp^+ dp^1 dp^2 \cdots dp^{d-1}}{(2\pi)^d}\frac{1}{p^+}\theta(p^+)\,,$$

where $p^+ = p^0 + p^3$ and $p^- = p^0 - p^3$.

This would correspond to performing the quantization on what is known as the light-front. For free particles, this is merely a relabeling of the quantum numbers, replacing P^0 with P^+. Later, when we consider canonical quantization, this change would correspond to imposing the canonical quantization condition at fixed $x^- = x^0 - x^3$, rather than at a fixed time x^0. As interesting as this approach is, we will not pursue it further in these lecture notes and will instead restrict ourselves to the standard quantization conditions in the following.

Finally, it is easy to show that the action of these operators on the one-particle states yields the following results:

$$\hat{H}^{(0)}\hat{a}^\dagger(\mathbf{k})\,|0\rangle = \omega_k\hat{a}^\dagger(\mathbf{k})\,|0\rangle\,,\ \hat{P}^i\hat{a}^\dagger(\mathbf{k})\,|0\rangle = k^i\hat{a}^\dagger(\mathbf{k})\,|0\rangle\,,\ \hat{N}\hat{a}^\dagger(\mathbf{k})\,|0\rangle = 1\cdot\hat{a}^\dagger(\mathbf{k})\,|0\rangle\,. \tag{3.35}$$

The attentive reader will have noticed that, while we have characterized all possible asymptotic states, we have not yet mentioned antiparticles. So, where are they? The short answer is that they haven't been overlooked. They are already included in the one-*particle* unitary irreducible representations we have obtained. Exactly where they appear will be addressed in the following chapters.

4 Scalar Free Fields

In the previous chapter, we focused on the one-particle unitary irreducible representations of the Poincaré group. This privileged working in momentum space, and it is the right attitude when focusing on the asymptotic states that appear in the S-matrix. Nevertheless, the interaction between particles is most often described in terms of fields in position space, since interaction terms are most easily written in position space. We therefore need to work out the relationship between momentum-space and position-space descriptions. In this chapter, we establish this relation for the case of scalar particles and KG fields.

4.1 Real KG Field

We consider the following Lagrangian density

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2}, \qquad L = \int d^d\mathbf{x}\, \mathcal{L}\,, \tag{4.1}$$

where $\phi(x) \in \mathbb{R}$.

Exercise

Deduce the E-L EoM associated with this Lagrangian.
Solution. One gets

$$\boxed{(\Box + m^2)\,\phi(t,\mathbf{x}) = 0}, \qquad \text{where} \quad \Box = \partial_\mu\partial^\mu = \partial_t^2 - \boldsymbol{\nabla}^2. \tag{4.2}$$

Hamiltonian Formalism (Towards Quantization)

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 \quad \Rightarrow \quad \pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x})} = \dot{\phi}(\mathbf{x}).$$

The Hamiltonian density then reads

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \pi^2 - \left(\frac{1}{2}\pi^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2\right) = \frac{1}{2}\left(\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right)\,.$$

This far, we have treated the fields as classical.

Canonical quantization rules

$$\begin{aligned} \{\phi(\mathbf{x}),\pi(\mathbf{y})\} &= \delta^{(d)}(\mathbf{x}-\mathbf{y}) \qquad \to \quad [\hat\phi(\mathbf{x}),\hat\pi(\mathbf{y})]=i\delta^{(d)}(\mathbf{x}-\mathbf{y}) \\ \{\hat\phi(\mathbf{x}),\hat\phi(\mathbf{y})\} &= \{\hat\pi(\mathbf{x}),\hat\pi(\mathbf{y})\}=0 \qquad [\hat\phi(\mathbf{x}),\hat\phi(\mathbf{y})]=[\hat\pi(\mathbf{x}),\hat\pi(\mathbf{y})]=0\,. \end{aligned} \tag{4.3}$$

The (quantized) Hamiltonian density reads

$$\hat{\mathcal{H}} = \frac{1}{2}\left(\hat\pi^2 + (\boldsymbol{\nabla}\hat\phi)^2 + m^2\hat\phi^2\right)\,.$$

Dynamics (Heisenberg Picture)

We now consider the time evolution of the fields

$$\hat\phi(t,\mathbf{x}) = e^{i\hat H t}\hat\phi(\mathbf{x})e^{-i\hat H t}, \qquad \hat\pi(t,\mathbf{x}) = e^{i\hat H t}\hat\pi(\mathbf{x})e^{-i\hat H t}. \tag{4.4}$$

These equalities can be transformed into differential equations, similarly to the procedure in Chapter 2 for NR fields. The outcome is nothing but the EoM in the Hamiltonian formalism. Let us obtain them and check that they also yield Eq. (4.2), but now in terms of quantum fields. For the time evolution of $\hat\phi(t,\mathbf{x})$, we have

$$i\frac{\partial}{\partial t}\hat\phi(t,\mathbf{x}) = \left[\hat\phi(t,\mathbf{x}),\hat H\right] = e^{i\hat H t}\left[\hat\phi(\mathbf{x}),\hat H\right]e^{-i\hat H t}. \tag{4.5}$$

For the commutator, we obtain

$$\begin{aligned} \left[\hat\phi(\mathbf{x}),\hat H\right] &= \left[\hat\phi(\mathbf{x}),\frac{1}{2}\int d^d\mathbf{y}\,\hat\pi^2(\mathbf{y})\right] \\ &= \frac{1}{2}\int d^d\mathbf{y}\left\{\hat\pi(\mathbf{x})\left[\hat\phi(\mathbf{x}),\hat\pi(\mathbf{y})\right]+\left[\hat\phi(\mathbf{x}),\hat\pi(\mathbf{y})\right]\hat\pi(\mathbf{y})\right\} \\ &= \frac{1}{2}\int d^d\mathbf{y}\left\{\hat\pi(\mathbf{x})i\delta^{(d)}(\mathbf{x}-\mathbf{y})+i\delta^{(d)}(\mathbf{x}-\mathbf{y})\hat\pi(\mathbf{y})\right\} = i\hat\pi(\mathbf{x}). \end{aligned} \tag{4.6}$$

Therefore, we obtain

$$i\frac{\partial}{\partial t}\hat\phi(t,\mathbf{x}) = i\hat\pi(\mathbf{x})\,. \tag{4.7}$$

We now generate the second equation:

$$i\frac{\partial}{\partial t}\hat\pi(t,\mathbf{x}) = \left[\hat\pi(t,\mathbf{x}),\hat H\right] = e^{i\hat H t}\left[\hat\pi(\mathbf{x}),\hat H\right]e^{-i\hat H t}.$$

To compute it, we need the following commutator:

$$\left[\hat\pi(\mathbf{x}),\hat H\right] = \frac{1}{2}\int d^d\mathbf{y}\left[\hat\pi(\mathbf{x}),(\boldsymbol{\nabla}\hat\phi(\mathbf{y}))^2+m^2\hat\phi^2(\mathbf{y})\right]\,.$$

The first term of this commutator yields

$$\frac{1}{2}m^2 \int d^d\mathbf{y} \left[\hat{\pi}(\mathbf{x}), \hat{\phi}^2(\mathbf{y})\right] = -im^2\hat{\phi}(\mathbf{x}),$$

and the second reads

$$\begin{aligned}
&\frac{1}{2}\int d^d\mathbf{y} \left[\hat{\pi}(\mathbf{x}), \left(\boldsymbol{\nabla}_{\mathbf{y}}\hat{\phi}(\mathbf{y})\right)^2\right] = \frac{1}{2}\int d^d\mathbf{y} \left[\hat{\pi}(\mathbf{x}), (\boldsymbol{\nabla}^i_{\mathbf{y}}\phi(\mathbf{y}))\right] \boldsymbol{\nabla}^i_{\mathbf{y}}\phi(\mathbf{y}) + \dots \\
&= -\frac{i}{2}\int d^d\mathbf{y}\, 2(\boldsymbol{\nabla}^i_{\mathbf{y}}\,\delta^{(d)}(\mathbf{x}-\mathbf{y}))\cdot \boldsymbol{\nabla}^i_{\mathbf{y}}\hat{\phi}(\mathbf{y}) = -\frac{i}{2}\int d^d\mathbf{y}\, 2\delta^{(d)}(\mathbf{x}-\mathbf{y})\boldsymbol{\nabla}^2_{\mathbf{y}}\hat{\phi}(\mathbf{y}) \\
&= i\boldsymbol{\nabla}^2_{\mathbf{x}}\hat{\phi}(\mathbf{x})\,.
\end{aligned}$$

So, overall

$$i\frac{\partial}{\partial t}\hat{\pi}(\mathbf{x},t) = +i\left(\boldsymbol{\nabla}^2_{\mathbf{x}} - m^2\right)\hat{\phi}(t,\mathbf{x}). \tag{4.8}$$

Combining Eqs. (4.7) and (4.8), we recover the KG equation:

$$\frac{\partial^2}{\partial t^2}\hat{\phi}(t,\mathbf{x}) = (\boldsymbol{\nabla}^2_{\mathbf{x}} - m^2)\hat{\phi}(t,\mathbf{x}) \quad \Rightarrow \quad (\Box + m^2)\hat{\phi}(x) = 0.$$

Solution to the KG equation

We now look for the solution to the KG equation. It is convenient to Fourier transform the KG field:

$$\phi(x) = \int d^Dk\, \tilde{\phi}(k)e^{-ik\cdot x},$$

then

$$(\Box + m^2)\phi(x) = \int d^Dk\, \tilde{\phi}(k)\left(-k^2 + m^2\right)e^{-ik\cdot x} = 0.$$

This implies

$$\tilde{\phi}(k) = 0 \quad \text{unless} \quad k^2 = k_0^2 - \mathbf{k}^2 = m^2 \quad \Rightarrow \quad \text{on-shell condition.}$$

Defining (see Eq. (3.34))

$$k_0 = \omega_k \equiv \sqrt{\mathbf{k}^2 + m^2}, \quad \phi(\omega_k, k) \propto a(\mathbf{k}), \quad d\tilde{k} = \frac{d^d\mathbf{k}}{(2\pi)^d 2\omega_k},$$

we can write the general solution as

$$\phi(x) = \int d\tilde{k}\left[a(\mathbf{k})e^{-ik\cdot x} + a^*(\mathbf{k})e^{ik\cdot x}\right],$$

where $a(\mathbf{k})$ is an arbitrary function.

Its quantum version (quantum field) is obtained merely transforming this expression into an operator:

$$\hat{\phi}(x) = \int d\tilde{k} \left[\hat{a}(\mathbf{k})e^{-ik\cdot x} + \hat{a}^\dagger(\mathbf{k})e^{ik\cdot x}\right].$$

$\hat{a}(\mathbf{k})$, $\hat{a}^\dagger(\mathbf{k})$ and their commutation relations

Exercise

Determine $\hat{a}(\mathbf{k})$ and $\hat{a}^\dagger(\mathbf{k})$ in terms of $\hat{\phi}(x)$ and its time derivative.
Hint: Use the following ansatz

$$\hat{a}(\mathbf{k}) \sim \int d^d\mathbf{x} e^{i\,k\cdot x}(i\dot{\hat{\phi}}(x) + \omega_k\hat{\phi}(x)),$$

where $k^0 = \omega_k = \sqrt{\mathbf{k}^2 + m^2}$. Work out the exact normalization factors by comparing with the mode expansion of the field. Finally, note that the result holds true for any x^0.

Exercise

Prove that

$$\begin{aligned}\left[\hat{a}(\mathbf{k}),\, \hat{a}^\dagger(\mathbf{k}')\right] &= 2\,\omega_k\,(2\pi)^d\,\delta^{(d)}\big(\mathbf{k}-\mathbf{k}'\big), \\ \left[\hat{a}(\mathbf{k}),\, \hat{a}(\mathbf{k}')\right] &= \left[\hat{a}^\dagger(\mathbf{k}),\, \hat{a}^\dagger(\mathbf{k}')\right] = 0\,.\end{aligned} \tag{4.9}$$

Hint: Use the canonical commutation relations $\left[\hat{\phi}(\mathbf{x}),\, \hat{\pi}(\mathbf{y})\right] = i\,\delta^{(3)}(\mathbf{x}-\mathbf{y})$, ..., together with the integral expressions (from the previous exercise) that define $\hat{a}(\mathbf{k})$ and $\hat{a}^\dagger(\mathbf{k})$.

Exercise

Show that

$$\hat{H} = \int d^d\mathbf{x}\frac{1}{2}\left[\dot{\hat{\phi}}^2 + \left(\boldsymbol{\nabla}\hat{\phi}\right)^2 + m^2\hat{\phi}\right] = \int d\tilde{k}\frac{\omega_k}{2}\left[\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})\right]. \tag{4.10}$$

Using this result and the commutation relations obtained in Eq. (4.9), we obtain

$$\hat{H} = \int d\tilde{k}\;\omega_k\;\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}') + \int d\tilde{k} w_k \frac{1}{2}\delta.$$

The term proportional to $\frac{1}{2}\delta$ is infinity but field-independent. Therefore, since we are only concerned with measuring energy differences (in particular with respect to the vacuum), we are not sensitive to this term.

The preceding results can be understood within the framework of the Fock Space for free massive spin-zero particles.[1] Recall that the vacuum $|0\rangle \equiv |0\rangle$ is defined by

$$\hat{a}(\mathbf{k})\,|0\rangle = 0 \quad \forall\,\mathbf{k}.$$

Acting with the creation operator on the vacuum produces a one-particle state

$$\hat{a}^\dagger(\mathbf{k})\,|0\rangle = |\mathbf{k}\rangle, \tag{4.11}$$

which is normalized in the following way

$$\langle\mathbf{k}'|\mathbf{k}\rangle = 2w_k(2\pi)^d\delta^{(d)}(\mathbf{k}'-\mathbf{k}) \tag{4.12}$$

If we consider $H\,|0\rangle$ or $H|\mathbf{k}\rangle$, we will obtain infinite quantities in both expressions but the energy difference is finite.

Similarly, multiple creation operators yield multiparticle states $|\mathbf{k}_1, \mathbf{k}_2, \ldots\rangle$. Again, in this case, energy differences are finite.

Normal ordering. For boson operators, the *normal ordering* of the product $\hat{a}(\mathbf{k})\,\hat{a}^\dagger(\mathbf{k})$ is defined as

$$:\hat{a}(\mathbf{k})\,\hat{a}^\dagger(\mathbf{k}): = \;\hat{a}^\dagger(\mathbf{k})\,\hat{a}(\mathbf{k}),$$

i.e. all creation operators move to the left and the annihilation operators to the right. Consequently, the *normal-ordered* Hamiltonian takes the form

$$:\hat{H}: = \int d\tilde{k}\,\omega_k\,\hat{a}^\dagger(\mathbf{k})\,\hat{a}(\mathbf{k}), \tag{4.13}$$

which eliminates the infinite vacuum-energy contribution. In this way, we do not have the problem of the vacuum energy being infinite.

Note that Eq. (4.13) is the Hamiltonian of free spin-zero particles given in Chapter 3, which did not have any infinity problems for the vacuum energy.

Conclusion

The canonical quantization of the Lagrangian of the real KG field generates the Hilbert space of free spin-zero massive (or massless) particles.

[1] To verify that it really represents a spin-zero particle and not (as bizarre as it may sound) a component of a non-zero spin particle, we must impose that the field transforms as a scalar (see Exercise 2 in Sec. 4.5).

Question. Is the KG Hamiltonian the only way to write a Hamiltonian for scalar particles in terms of fields? Why not just work with $\hat{\phi}^{(+)}(x)/\hat{\phi}^{(-)}(x)$?

In principle, it is possible to write the Hamiltonian using

$$\hat{\phi}^{(+)}(x) = \int d\tilde{k}\, \hat{a}(\mathbf{k}) e^{-ik\cdot x}, \qquad \hat{\phi}^{(-)}(x) = \hat{\phi}^{(+)\dagger}(x) = \int d\tilde{k}\, \hat{a}^\dagger(\mathbf{k}) e^{ik\cdot x}.$$

as building blocks.

$$\hat{H} = \int d^d\mathbf{x} \left[\dot{\hat{\phi}}^{(-)}(x)\dot{\hat{\phi}}^{(+)}(x) + \left(\boldsymbol{\nabla}\hat{\phi}^{(-)}(x)\right)\left(\boldsymbol{\nabla}\hat{\phi}^{(+)}(x)\right) + m^2\hat{\phi}^{(-)}(x)\hat{\phi}^{(+)}(x)\right],$$

gives the right $\hat{H}$ without any infinite contribution.

There is also another possibility:

$$\hat{H} = \int d^d\mathbf{x} \left[\hat{\phi}^{(-)}_{\rm NR}(x)\sqrt{-\boldsymbol{\nabla}^2 + m^2}\, \hat{\phi}^{(+)}_{\rm NR}(x)\right], \qquad (4.14)$$

using the following fields with NR normalization

$$\hat{\phi}^{(+)}_{\rm NR}(x) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\hat{a}(\mathbf{k})e^{-ikx}, \qquad \hat{\phi}^{(-)}_{\rm NR}(x) = \hat{\phi}^{(+)\dagger}_{\rm NR}(x) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\hat{a}^\dagger_{\rm NR}(\mathbf{k})e^{+ikx},$$

$$\left[\hat{a}_{\rm NR}(\mathbf{k}),\, \hat{a}^\dagger_{\rm NR}(\mathbf{k}')\right] = (2\pi)^d\, \delta^{(d)}\left(\mathbf{k}-\mathbf{k}'\right) \qquad (4.15)$$

$$\{\hat{\phi}^{(+)}_{\rm NR}(x^0,\mathbf{x}), \hat{\phi}^{(-)}_{\rm NR}(x^0,\mathbf{y})\} = \delta^{(d)}(\mathbf{x}-\mathbf{y}). \qquad (4.16)$$

This last option arises from the alternative Lagrangian

$$\mathcal{L} = \phi^{(-)}_{\rm NR}(x)\left(i\partial_0 - \sqrt{-\boldsymbol{\nabla}^2 + m^2}\right)\phi^{(+)}_{\rm NR}(x). \qquad (4.17)$$

This Lagrangian can be related to the original KG Lagrangian using Foldy-Wouthuysen transformations.[2]

There is no problem with these constructions at the level of free particles. The problem arises when one attempts to introduce interactions in relativistic QFTs in terms of $\phi^{(+)}$ and $\phi^{(-)}$ independently (actually it is possible but not natural, as the natural object that appears in the interaction term will be $\phi(x)$, the KG field).

[2]The application of Foldy-Wouthuysen transformations to relativistic spin-$\frac{1}{2}$ particles can be found in [7].

4.1.1 Propagators

Feynman propagators are basic objects for the computation of S-matrix elements. Let us discuss the general structure of propagators for KG fields.

We first compute the commutator between fields at different times (thus departing from the canonical quantization conditions):

$$\left[\hat{\phi}(t,\mathbf{x}),\hat{\phi}(t',\mathbf{y})\right] = \left[\hat{\phi}^{(+)}(x),\hat{\phi}^{(-)}(y)\right] + \left[\hat{\phi}^{(-)}(x),\hat{\phi}^{(+)}(y)\right] \tag{4.18}$$

$$= \int d\tilde{k} \int d\tilde{k}' \left[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k}')\right] e^{-ik\cdot x} e^{ik'\cdot y} + \dots \tag{4.19}$$

$$= \int d\tilde{k}\, e^{-ik\cdot(x-y)} + \dots \equiv i\Delta^+(x-y) + i\Delta^-(x-y) \equiv i\Delta(x-y), \tag{4.20}$$

where $x \equiv (t,\mathbf{x})$; $y \equiv (t',\mathbf{y})$;

$$\int d\tilde{e}^{-ik(x-y)} \equiv i\Delta^+(x-y)\,,$$

and we have the conjugation property

$$\Delta^+(x-y) = -\Delta^-(y-x), \quad \text{and} \quad \Delta(x) = -\Delta(-x).$$

Computation of $\Delta(x)$

We now derive an explicit expression for $\Delta(x)$.

$$\begin{aligned} \Delta(x) &= -i\int d\tilde{k}\left(e^{-ik\cdot x} - e^{ik\cdot x}\right) \\ &= -2\int d\tilde{k}\sin(k\cdot x) = -\frac{1}{(2\pi)^d}\int \frac{d^d\mathbf{k}}{\omega_k}\sin(k\cdot x), \end{aligned} \tag{4.21}$$

where $k\cdot x = \omega_k t - \mathbf{k}\cdot\mathbf{x}$. Then

$$\Delta(x) = \frac{m}{4\pi\sqrt{\lambda}}\epsilon(x^0)\theta(\lambda)J_1(m\sqrt{\lambda}) - \frac{1}{2\pi}\epsilon(x^0)\delta(\lambda), \quad \text{where } \lambda = x^2 = (x^0)^2 - |\mathbf{x}|^2,$$

where

$$\epsilon(n) = \theta(n) - \theta(-n)\,.$$

From this result, it is evident that $\Delta(x)$ behaves as a scalar under Poincaré transformations.

Exercise

Show that

$$[\phi(x), \phi(y)] = 0 \quad \text{for space-like separation vectors, i.e. for } (x-y)^2 < 0\,.$$

This property is called **microcausality**. It holds in any reference frame. Note that, for space-like vectors, there is always a frame in which $x^0 = y^0$.

Discussion

If we promote $\hat{\phi}(x)$ to be an observable, we may be interested in computing

$$\langle 0| \left[\hat{\phi}(x)\hat{\phi}(y) - \hat{\phi}(y)\hat{\phi}(x)\right] |0\rangle \ .$$

As expected, this is zero for space-like distances — the ordering of measurements should not matter at space-like distances. The problem with this argument is that fields cannot always be promoted to observables. A necessary condition is that they should be Hermitian, something not fulfilled by fermion fields, for instance. One may also think that the same problem appears for complex fields, but those can typically be rewritten as combinations of real fields.

On the other hand, note that

$$\left[\hat{\phi}^{(+)}, \hat{\phi}^{(-)}\right] \neq 0 \quad \text{for space-like separation.}$$

If microcausality is to play a role in our construction of interacting relativistic theories, this result suggests that $\phi^{(+)}/\phi^{(-)}$ alone are not the optimal building blocks to write such a theory. Instead, one should use a specific linear combination of them.

Towards a more covariant representation of the propagators

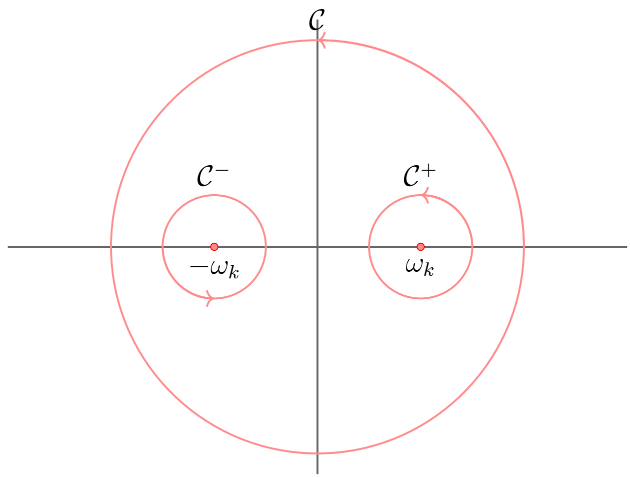
We can rewrite $\Delta^{\pm}(x)$ in the following way:

$$\Delta^{\pm}(x) = -\frac{1}{(2\pi)^D}\int_{C^{\pm}} d^D k \, \frac{e^{-ik\cdot x}}{k^2 - m^2}, \tag{4.22}$$

where k^0 is a complex variable running along the $C^{\pm}$ path. The denominator can be written as $k^2 - m^2 = k_0^2 - \mathbf{k} - m^2 = k_0^2 - \omega_k^2 = (k_0 - \omega_k)(k_0 + \omega_k)$. Therefore, $k_0 = \pm\omega_k$ are singularities. Overall, the integrand is analytic throughout the complex plane except at $k^0 = \pm w_k$, where the integrand has simple poles.

The k^0 integration paths for $C^{\pm}$ and C are drawn below:

Paths of Integration



It is easy to see that $\Delta^{\pm}(x)$ can be written as in Eq. (4.22) by using

$$\oint_{C^+} dk^0 \left(\frac{e^{-ik^0 t}}{2\omega_k (k_0 - \omega_k)} + \text{non-singular terms} \right) = 2\pi i \, \frac{e^{-i\omega_k t}}{2\omega_k} \,, \tag{4.23}$$

and alike. This confirms that Eq. (4.22) is correct. We can perform a similar analysis for $\Delta^-(x)$.

We can also obtain

$$\Delta(x) = -\frac{1}{(2\pi)^D} \int_C d^D k \, \frac{e^{-ik\cdot x}}{k^2 - m^2} \,. \tag{4.24}$$

We will later use the following equalities:

$$\begin{aligned} i\Delta^+(x-y) &= \left[\hat{\phi}^{(+)}(x), \hat{\phi}^{(-)}(y)\right] = \langle 0| \left[\hat{\phi}^{(+)}(x), \hat{\phi}^{(-)}(y)\right] |0\rangle \\ &= \langle 0|\hat{\phi}^{(+)}(x)\hat{\phi}^{(-)}(y)|0\rangle = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \,. \end{aligned} \tag{4.25}$$

Note that $\left[\hat{\phi}^{(+)}, \hat{\phi}^{(-)}\right] = $ *c-number* and can be related with the expectation value in the vacuum of two KG fields at different space-time positions, i.e. with its correlation.

Note also that

$$(\Box_x + m^2)\Delta^{\pm}(x-y) = 0,$$

and, likewise, for any linear combination of them. Therefore, these are not yet Green's functions, although they will be related to them.

4.1.2 Feynman Propagator

Unlike the propagators considered so far, the Feynman propagator is the quantity that appears in actual computations of physical processes. We begin by defining the time-ordering operation (for bosonic operators)

$$T\left\{\hat{\phi}(x)\hat{\phi}(x')\right\} = \begin{cases} \hat{\phi}(x)\hat{\phi}(x') & t > t', \\ \hat{\phi}(x')\hat{\phi}(x) & t' > t. \end{cases}$$

Using the Heaviside step function, the above operation can be written as:

$$T\left[\hat{\phi}(x)\hat{\phi}(x')\right] = \theta(t-t')\hat{\phi}(x)\hat{\phi}(x') + \theta(t'-t)\hat{\phi}(x')\hat{\phi}(x)\,.$$

We define the **Feynman Propagator** as

$$\text{F.P.} \equiv i\Delta_F(x-x') = \langle 0|T\left\{\hat{\phi}(x)\hat{\phi}(x')\right\}|0\rangle\,.$$

So,

$$\Delta_F(x) = \theta(t)\Delta^+(x) - \Delta^-(x)\theta(-t)\,.$$

Exercise

Determine $(\Box_x + m^2)\Delta_F(x)$
Solution: We first compute

$$\begin{aligned}(\Box_x + m^2)(\theta(t)\Delta^+) &= (\partial_t\partial_t - \boldsymbol{\nabla}^2 + m^2)(\theta\Delta^+) \\ &= \partial_t^2(\theta\Delta^+) + \theta(-\boldsymbol{\nabla}^2 + m^2)\Delta^+ = \partial_t^2(\theta\Delta^+) - \cancel{\theta\partial_t^2\Delta^+}\end{aligned}$$

Now note that

$$\partial(\theta\Delta^+) = \partial_t(\theta)\Delta^+ + \theta\partial_t\Delta^+ = \delta(t)\Delta^+ + \theta(t)(\partial_t\Delta^+)$$
$$\downarrow$$
$$\partial_t^2(\theta\Delta^+) = \partial_t\left[\delta(t)\Delta^+ + \theta(t)(\partial_t\Delta^+)\right] = \partial_t(\delta(t)\Delta^+) + \delta(t)\partial_t\Delta^+ + \cancel{\theta\partial_t^2\Delta^+}$$
$$\downarrow$$
$$\begin{aligned}\delta(t)\partial_t\Delta^+ = \delta(t)\left[\partial_t\left(-i\int d\tilde{k}e^{-ikx}\right)\right] &= \delta(t)\left[-\int \tfrac{d^d\mathbf{k}}{(2\pi)^d}\,\tfrac{1}{2\omega_k}\omega_k e^{-ikx}\right] \\ &= -\tfrac{1}{2}\delta(t)\delta^{(d)}(\mathbf{x}) = -\tfrac{1}{2}\delta^{(D)}(x)\end{aligned}$$

$$\Rightarrow (\Box_x + m^2)(\theta(t)\Delta^+) = \partial_t\left(\delta(t)\Delta^+\right) - \frac{1}{2}\delta^{(D)}(x).$$

Exercise (Cont.)

We leave as an exercise to the reader to show that

$$(\Box + m^2)(\theta(-t)\Delta^-) = \partial_t(-\delta\Delta^-) + \frac{1}{2}\delta^{(D)}(x) \tag{4.26}$$

Therefore, we have

$$(\Box + m^2)\Delta_F(x) = \partial_t\left(\delta(t)(\Delta^+ + \Delta^-)\right) - \delta^{(D)}(x). \tag{4.27}$$

Let us focus on the first term:

$$\begin{aligned}\delta(t)\Delta &= \delta(t)\left(-\frac{1}{(2\pi)^d}\right)\int \frac{d^d\mathbf{k}}{2\omega_k}\sin\left(k_0 t - \mathbf{k}\cdot\mathbf{x}\right) \\ &\overset{t=0}{=} \delta(t)\left(-\frac{1}{(2\pi)^d}\right)\int_{-\infty}^{\infty} \frac{d^d\mathbf{k}}{2\omega_k}\sin\left(-\mathbf{k}\cdot\mathbf{x}\right) \overset{\sin\to\text{odd}}{=} 0.\end{aligned} \tag{4.28}$$

Therefore,

$$(\Box + m^2)\Delta_F(x) = -\delta^{(D)}(x)\,. \tag{4.29}$$

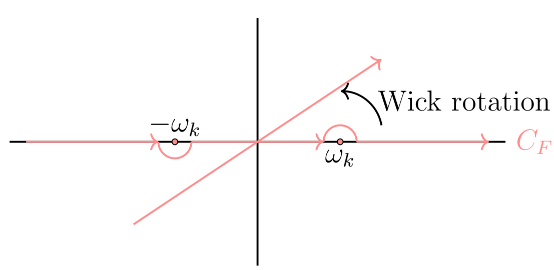
This means that $\Delta_F(x)$ is a Green's function. Note that this result holds in any reference frame.

Covariant expression of the Feynman propagator

We write[3]

$$\Delta_F(x) = \frac{1}{(2\pi)^D}\int_{C_F} d^D k\,\frac{e^{-ik\cdot x}}{k^2 - m^2} = \frac{1}{(2\pi)^D}\int d^d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{x}}\int_{C_F} dk^0\,\frac{e^{-ik_0\cdot x}}{k_0^2 - \omega_k^2}. \tag{4.30}$$

Note again that k^0 is a complex variable. The path C_F can be found in the figure below[4]



[3]Note the overall change of sign with respect to the definition of $\Delta(x)$.

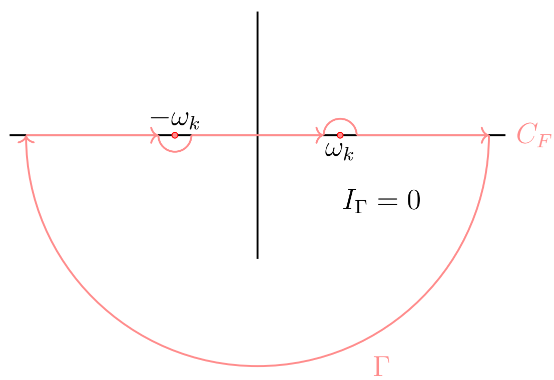
[4]Wick rotation profits from the fact that we can deform the integration path as long as we do not cross any singularity, and the contribution at infinity can be neglected. It is then customary to rotate 90º and make k^0 purely imaginary.

As in the case of $\Delta(x)$, the integrand has two simple poles, located at $k^0 = \pm w_k$. However, unlike in the case of $\Delta(x)$, the integration path in the complex plane is open. Therefore, we cannot directly apply Cauchy's theorem. We must first close the contour. We will then use Jordan's Lemma.[5] To do so, we must distinguish between the case $t > 0$ and $t < 0$.

Case $t > 0$: Let $i\text{Im}[k_0] = ib$, $\Rightarrow -ik_0 t = bt$. Therefore, we need $b < 0$ to apply Jordan's Lemma and neglect the semicircle contribution to the integral. Consequently, we close the integration path in the lower half-plane. This captures the pole at $k^0 = \omega_k$, so

$$I_F + I_\Gamma = -(2\pi i)\sum \text{Res} f(z) \quad \Rightarrow \quad \Delta_F(x) = \Delta^+(x) \ \text{ for } t > 0\,,$$

since we can deform the path to make it equal to $-C_+$. We show the integration path below.



Exercise

Case $t < 0$: We leave as an exercise to the reader to verify that

$$\Delta_F(x) = \Delta^-(x)\ .$$

Also note that

$$\begin{aligned}(\Box + m^2)\Delta_F(x) &= \frac{1}{(2\pi)^D}\int_{C_F} d^D x\, \frac{(\Box + m^2)e^{-ikx}}{k^2 - m^2} \\ &= -\frac{1}{(2\pi)^D}\int_{C_F} d^D x\, e^{-ikx} = -\delta^{(D)}(x),\end{aligned} \tag{4.31}$$

where in the last equality we have used that C_F lies almost on the real axis and the Fourier transform representation of the Dirac delta in distribution theory.

[5] Note that, leaving aside the exponential factor, the integrand decays fast enough at infinity to be eligible for applying Jordan's Lemma.

General solution to the KG equation with external sources

$$(\Box+m^2)F(x) = -f(x) = -\int d^Dy\, f(y)\delta^{(D)}(x-y) = \int d^Dy\, f(y)(\Box+m^2)\Delta_F(x-y)$$

$$\boxed{\Rightarrow F(x) = \int d^Dy\, f(y)\, \Delta_F(x-y) + A\Delta^+(x) + B\Delta^-(x)\,,}$$

Particular solution   Homogeneous solution

where A and B are arbitrary constants.

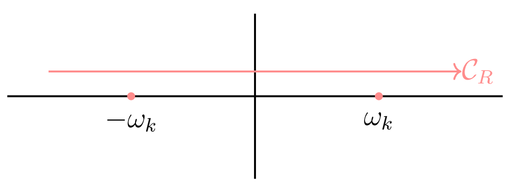
Other possibilities: Retarded and Advanced

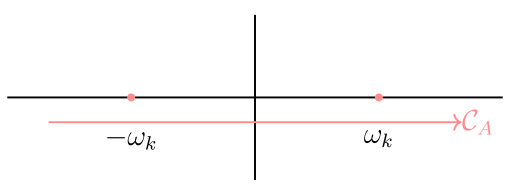
We can also define the **retarded** and **advanced** propagators (among other possibilities, see [2], for instance)

$$\Delta_{A,R}(x) = \frac{1}{(2\pi)^D}\int_{\mathcal{C}_{A,R}} d^Dk\, \frac{e^{-ikx}}{k^2-m^2},$$

where

$$t>0 \rightarrow \begin{cases} \Delta_R = \Delta, \\ \Delta_A = 0. \end{cases} \qquad t<0 \rightarrow \begin{cases} \Delta_R = 0, \\ \Delta_A = -\Delta. \end{cases} \tag{4.32}$$





We obtain

$$\boxed{\Delta_R = \theta(t)\,\Delta, \qquad \Delta_A = -\theta(-t)\,\Delta \quad \Rightarrow \quad \Delta_R - \Delta_A = \Delta\,.}$$

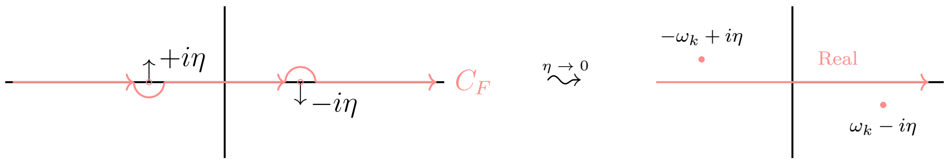
Exercise

Show that

$$(\Box_x + m^2)\,\Delta_R(x-y) = -\delta^{(D)}(x-y)\,.$$

Alternative expression for the Feynman propagator

We can infinitesimally move the position of the poles as follows



This ensures that we can treat the four components of k^μ as real variables and rewrite the Feynman propagator in the following way:

$$\Delta_F(x) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} d^D k \, \frac{e^{-ikx}}{k^2 - m^2 + i\eta} \,, \tag{4.33}$$

where the $\eta \to 0$ limit is understood. This is the standard expression used in perturbative computations of the S-matrix. Let us end by emphasizing that k^0 here cannot be interpreted as the physical energy of the particle, since $k^0 \neq w_k$.

4.2 Symmetries

We have chosen to postpone the discussion of symmetries and Noether's theorem until this point, even though part of it could have been addressed at the classical level in Sec. 1.3. The main reason for doing so is that now we can also develop the quantum version in parallel.

We will use the real KG field as a template, although the methodology will be general.

Field Redefinition (Transformation)

We consider a Lagrangian density $\mathcal{L}(\phi)$ (for simplicity we omit derivatives of the fields, or any explicit x dependence). We are always free to make a field redefinition (a change of "coordinates") as follows:

$$\phi' = \phi + \delta\phi \,.$$

We define the Lagrangian associated with the new field as

$$\mathcal{L}'(\phi') \equiv \mathcal{L}(\phi(\phi')) \quad \Rightarrow \quad S = \int d^D x \, \mathcal{L}(\phi) = \int d^D x \, \mathcal{L}'(\phi') \,.$$

If ϕ_0 is a solution of the EoM of $\mathcal{L}(\phi)$, then $\longrightarrow$ ϕ'_0 is a solution of $\mathcal{L}'(\phi')$.

For Lagrangians that depend on, at most, first-order derivatives,[6] we define the following shorthand notation for the E-L operator:

$$[\mathcal{L}]_\phi = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \,.$$

We then have ($\delta S = 0$)

$$[\mathcal{L}]_{\phi_0} = 0 \iff [\mathcal{L}']_{\phi_0'} = 0,$$

Nevertheless, not all field redefinitions are considered to be symmetries. We provide a definition of symmetry below.

Definition

A transformation $\delta\phi$ is a **symmetry** of the action if for every ϕ solution of the EoM,

$$[\mathcal{L}]_\phi = 0 \quad \Rightarrow \quad [\mathcal{L}]_{\phi+\delta\phi} = 0.$$

We emphasize that the Lagrangian has not changed.

4.2.1 Noether's Theorem (for continuous symmetries: Lie groups)

Noether's theorem provides a powerful link between continuous symmetries and conserved quantities:

- It gives a *sufficient condition* to determine whether a field transformation is a symmetry.
- It also determines the associated *conserved quantities* (also called *charges*) related to these symmetries. These conserved quantities are constant in time. In some cases, but not always, they will commute with the Hamiltonian. Recall that this last condition is required for obtaining a complete set of compatible observables. This is not the case with all generators of the Poincaré group. In practice, this means that such operators will exhibit some explicit time dependence.

Let us consider a Lagrangian density $\mathcal{L}(\phi(x))$ and the associated action

$$S = \int d^D x \, \mathcal{L}(\phi).$$

We consider a transformation $\phi \to \phi' = \phi + \delta\phi$ and the new action

$$S' = \int d^D x \, \mathcal{L}(\phi').$$

[6] Otherwise, we should generalize this definition.

We minimize the action in both cases

$$\delta S = 0, \qquad \delta S' = 0.$$

Using

$$S' = \int d^D x \, [\mathcal{L}(\phi') - \mathcal{L}(\phi)] + \int d^D x \, \mathcal{L}(\phi).$$

If the Lagrangian changes by a total derivative

$$\mathcal{L}(\phi') - \mathcal{L}(\phi) = \partial_\mu \Omega^\mu,$$

and the fields vanish at infinity, then

$$\int d^D x \, [\mathcal{L}(\phi') - \mathcal{L}(\phi)] = 0 \,, \tag{4.34}$$

and both actions yield the same EoM. This is Noether's result:

Noether's Theorem 1st part

A sufficient condition for a field transformation $\delta\phi$ to be a symmetry is

$$\mathcal{L}(\phi') - \mathcal{L}(\phi) = \partial_\mu \Omega^\mu \,. \tag{4.35}$$

The second part of Noether's theorem states that there are associated conserved currents for every continuous symmetry. Let us derive them. We expand the change in the Lagrangian to first order under a small field transformation:

$$\begin{aligned}
\mathcal{L}(\phi') - \mathcal{L}(\phi) = \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) \\
&= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi \\
&= \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) \\
&\overset{(4.35)}{=} \partial_\mu \Omega^\mu,
\end{aligned}$$

where we have used $\delta(\partial_\mu\phi) = \partial_\mu\delta\phi$. We emphasize that ∂_μ in the expressions above is a total derivative (acting on the fields and on the explicit x dependence). All the above equalities are identities except the final one, which is the condition for the field transformation to be a symmetry. Overall, we have

$$[\mathcal{L}]_\phi \delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - \Omega^\mu\right) = 0.$$

If ϕ satisfies the EoM, i.e., $[\mathcal{L}]_\phi = 0$, then we get

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \Omega^\mu \right) = 0\,,$$

and

Definition

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \Omega^\mu, \tag{4.36}$$

is a conserved current. If the fields are solution of the EoM, we have that

$$\partial_\mu J^\mu \Big|_{\phi-\text{sol. EoM}} = 0\,. \tag{4.37}$$

In the general case of fields with an extra quantum number A (which could either be internal or associated with the Poincaré symmetry), one has

$$\boxed{J^\mu \equiv \sum_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \delta\phi_A - \Omega^\mu.} \tag{4.38}$$

From this result, and for fields that are zero at (spatial) infinity, we can also write the corresponding **Conserved charges**:

$$\boxed{Q \equiv \int d^d\mathbf{x}\, J^0(t, \mathbf{x}),}$$

i.e., they are time-independent:

$$\frac{dQ}{dt} = \int d^d\mathbf{x} \frac{\partial J^0}{\partial t} = \int d^d\mathbf{x}\, \partial_i J^i = -\int_{\partial V} J^i d\sigma^i \underset{\phi(\infty)\to 0}{=} 0.$$

Note that the conserved charges are defined up to arbitrary multiplicative constants.

Discussion about Total Derivative Terms in the Lagrangian[7]*

- **CM**

  We first discuss the case in CM. We consider a Lagrangian modified by a total time derivative term

$$L'(q_j, \dot{q}_j, t) = L(q_j, \dot{q}_j, t) + \frac{d}{dt} F(q_j, t)\,.$$

[7]This discussion is not fundamental for the purposes of this book, and could be skipped in a first reading.

Such a term does not affect the EoM. Let us check this out.

$$G \equiv \frac{d}{dt}F(q_j,t) = \frac{\partial}{\partial t}F(q_j,t) + \sum \dot{q}_j \frac{\partial}{\partial q_j}F(q_j,t).$$

Since

$$\frac{\partial G}{\partial \dot{q}_j} = \frac{\partial}{\partial q_j}F(q_j,t)\,,$$

we get that the E-L EoM

$$\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = 0,$$

applied to G is nothing but an identity:

$$\frac{d}{dt}\frac{\partial F}{\partial q_j} - \frac{\partial}{\partial q_j}\frac{dF}{dt} = 0 \qquad \text{(due to the commutativity of the derivatives).}$$

- **Field theory**

  We now consider

$$\Omega^\mu = \Omega^\mu(x,\phi).$$

  The generalization of a total derivative is

$$G \equiv \partial^T_\mu \Omega^\mu(x,\phi) = \partial_\mu \Omega^\mu(x,\phi) + \frac{\partial \Omega^\mu}{\partial \phi}\,\partial_\mu \phi.$$

  Since

$$\frac{\partial G}{\partial(\partial_\mu \phi)} = \frac{\partial \Omega^\mu}{\partial \phi}\,,$$

  we get that the E-L EoM,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^T_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0\,,$$

  applied to G is, again, nothing but an identity:

$$\frac{\partial}{\partial \phi}\left(\partial^T_\nu \Omega^\nu\right) - \partial^T_\mu \left(\frac{\partial \Omega^\mu}{\partial \phi}\right) = \frac{\partial}{\partial \phi}\frac{\partial^T}{\partial x^\nu}\Omega^\nu - \frac{\partial^T}{\partial x^\nu}\frac{\partial}{\partial \phi}\Omega^\nu = 0,$$

  due to the commutativity of the derivatives.

- One can write a more general expression for Ω_μ:

$$\Omega_\mu \to \Omega_\mu(x,\phi,\phi_\nu,\ldots),$$

  but then the E–L EoM must be modified accordingly.

4.2.2 Translations

Let us apply Noether's theorem to translations. A space-time translation acts as

$$x^\mu \to x'^\mu = x^\mu + \delta a^\mu .$$

For scalar fields, the invariance condition is

$$\phi'(x') = \phi(x),$$

which means that both observers assign the same numerical value to the field $\forall\, x$. In the quantum theory, the equality should be understood at the level of operators (one obtains the same numerical values for all possible matrix elements).

The invariance condition can be written in the following way

$$\phi'(x) = \phi(x - \delta a)\,, \tag{4.39}$$

which is more convenient for us. Expanding to first order in δa, the variation of the field is (see also Sec. 1.4.4)

$$\delta\phi(x) = \phi'(x) - \phi(x) = \phi'(x' - \delta a) - \phi(x) = \phi'(x') - \delta a^\mu \frac{\partial \phi}{\partial x^\mu} - \phi(x) = -\delta a^\mu \partial_\mu \phi\,.$$

We now consider variations for one of the specific coordinates

$$\boxed{\delta_{[\mu]}\phi = -\left(\partial_{[\mu]}\phi\right)\, \delta a^{[\mu]}\,,} \tag{4.40}$$

where there is no summation convention when we write $[\mu]$, i.e. μ is fixed: We perform a translation for each direction.

Now, let us compute the variation of the Lagrangian density $\mathcal{L}(\phi, \phi_{,\mu}; x)$. We obtain

$$\delta_{[\mu]}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta_{[\mu]}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)}\partial_\sigma \delta_{[\mu]}\phi.$$

Using the variation $\delta\phi = -\delta a^{[\nu]}\partial_{[\nu]}\phi$, we get

$$\begin{aligned} \delta_{[\mu]}\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\left(-\partial_{[\mu]}\phi\,\delta a^{[\mu]}\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)}\left(\partial_\sigma\left(-\partial_{[\mu]}\phi\right)\delta a^{[\mu]}\right) \\ &= -\left(\frac{\partial \mathcal{L}}{\partial \phi}\partial_{[\mu]}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma \phi)}\partial_\sigma\left(\partial_{[\mu]}\phi\right)\right)\delta a^{[\mu]} = -\left(\partial_\mu^T \mathcal{L}\right)\delta a^{[\mu]}. \end{aligned} \tag{4.41}$$

If $\mathcal{L}$ has NO explicit x-dependence: $\mathcal{L} = \mathcal{L}(\phi, \phi_{,\mu})$

With the label T in ∂^T_μ, we want to emphasize that it is a total derivative, which can be rewritten in the following way:

$$\delta_{[\mu]}\mathcal{L} = -\partial^T_\sigma \left(\mathcal{L}\delta^\sigma_{[\mu]}\delta a^{[\mu]}\right), \tag{4.42}$$

so it satisfies Noether's condition for symmetry: $\delta\mathcal{L} = \partial_\mu\Omega^\mu$.

Let's now determine the current. The above expression can also be written as:

$$\delta_{[\mu]}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta_{[\mu]}\phi + \frac{\partial\mathcal{L}}{\partial\phi_{\prime\nu}}\partial_\nu\delta_{[\mu]}\phi = [\mathcal{L}]\delta_{[\mu]}\phi + \partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial\phi_{\prime\sigma}}\delta_{[\mu]}\phi\right). \tag{4.43}$$

By combining Eqs. (4.42) and (4.43), we obtain

$$[\mathcal{L}]\delta_{[\mu]}\phi + \partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial\phi_{\prime\sigma}}\delta_{[\mu]}\phi + \mathcal{L}\delta^\sigma_{[\mu]}\delta a^{[\mu]}\right) = 0, \tag{4.44}$$

and the conserved current reads

$$J^\sigma_{[\mu]} = \left(-\delta_{[\mu]}\phi\delta a^{[\mu]}\right)\frac{\partial\mathcal{L}}{\partial\phi_{\prime\sigma}} + \mathcal{L}\delta^\sigma_{[\mu]}\delta a^{[\mu]} = -\left(\frac{\partial\mathcal{L}}{\partial\phi_{\prime\sigma}}\delta_{[\mu]}\phi - \mathcal{L}\delta^\sigma_{[\mu]}\right)\delta a^{[\mu]}. \tag{4.45}$$

Remark: This current can be multiplied by a constant, if desired, without breaking the conservation condition.

Definition

Energy-momentum tensor (Noether's current associated with space-time translations)[a]

$$T^{\sigma\mu} = \frac{\partial\mathcal{L}}{\partial\phi_{\prime\sigma}}\partial^\mu\phi - \eta^{\sigma\mu}\mathcal{L}.$$

This expression holds true for any arbitrary interacting theory made of KG fields with a Lagrangian density of the form: $\mathcal{L}(\phi, \phi_{\prime\mu})$.

[a] It transforms as a tensor with two indices under Lorentz transformations.

Generator of Translations

From the energy-momentum tensor $T^{\mu\nu}$, we obtain

$$P^\mu = \int d^d\mathbf{x}\, T^{0\mu}.$$

These are the conserved charges associated with translational symmetry.

In the case of a free scalar field, we have

$$T^{\mu\nu} = (\partial^\mu\phi)(\partial^\nu\phi) - g^{\mu\nu}\left[\frac{1}{2}(\partial_\lambda\phi)(\partial^\lambda\phi) - \frac{1}{2}m^2\phi^2\right].$$

It is easy to see that $P^0 = H^0$. It can also be written in terms of creation and annihilation operators (see Eq. (4.10)).

For the spatial momentum operator, we have

$$\mathbf{P}^i = \int d^d\mathbf{x}(\partial^0\phi)(\partial^i\phi)\,.$$

Exercise

Show that

$$\hat{\mathbf{P}} = -\int d^d\mathbf{x}\,\dot{\hat{\phi}}(\mathbf{x})\boldsymbol{\nabla}\hat{\phi}(\mathbf{x}) = \int d\tilde{k}\;\mathbf{k}\,\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}). \tag{4.46}$$

Note that in this case, unlike for P^0, there is no need for normal ordering to get this result.

Exercise

Show that P^μ are the generators of translations.

Solution

We first consider how the creation operator transforms under translations

$$\hat{a}^\dagger(\mathbf{k},\lambda) \equiv e^{i\lambda_\mu\hat{P}^\mu}\hat{a}^\dagger(\mathbf{k})e^{-i\lambda_\mu\hat{P}^\mu} =? \tag{4.47}$$

We expect this quantity to be proportional to $\hat{a}^\dagger(\mathbf{k})$ up to, maybe, a phase factor, since the state does not change under these transformations. Similarly to the time evolution discussed previously in the NR case or for KG, $\hat{a}^\dagger(\mathbf{k},\lambda)$ can be obtained by generating a differential equation:

$$i\frac{\partial}{\partial_{\lambda_\mu}}\hat{a}^\dagger(\mathbf{k},\lambda) = \left[\hat{a}^\dagger(\mathbf{k},\lambda),\hat{P}^\mu\right] = e^{i\lambda_\mu\hat{P}^\mu}\left[\hat{a}^\dagger(\mathbf{k}),\hat{P}^\mu\right]e^{-i\lambda_\mu\hat{P}^\mu} = -k^\mu\hat{a}^\dagger(\mathbf{k},\lambda)\,, \tag{4.48}$$

where we have used

$$\left[\hat{P}^\mu,\hat{a}^\dagger(\mathbf{k})\right] = k^\mu\hat{a}^\dagger(\mathbf{k})\,. \tag{4.49}$$

Solving the differential equation, we obtain

$$\begin{aligned}\hat{a}^\dagger(\mathbf{k},\lambda) &= e^{i\lambda_\mu\hat{P}^\mu}\hat{a}^\dagger(\mathbf{k})e^{-i\lambda_\mu\hat{P}^\mu} = e^{+i\lambda_\mu k^\mu}\hat{a}^\dagger(\mathbf{k}),\\ \hat{a}(\mathbf{k},\lambda) &= e^{i\lambda_\mu\hat{P}^\mu}\hat{a}(\mathbf{k})e^{-i\lambda_\mu\hat{P}^\mu} = e^{-i\lambda_\mu k^\mu}\hat{a}(\mathbf{k}).\end{aligned} \tag{4.50}$$

Exercise (Cont.)

The creation and annihilation operators acquire a momentum-dependent phase under translations:

$$\boxed{\hat{a}^\dagger(\mathbf{k}) \to e^{+i\lambda_\mu k^\mu}\hat{a}^\dagger(\mathbf{k}), \qquad \hat{a}(\mathbf{k}) \to e^{-i\lambda_\mu k^\mu}\hat{a}(\mathbf{k}),}$$

$$\Rightarrow \hat{a}^\dagger(\mathbf{k})|0\rangle \to e^{+i\lambda_\mu k^\mu}\hat{a}^\dagger(\mathbf{k})|0\rangle\,. \tag{4.51}$$

We now check how the field operator itself transforms. Given

$$\hat{\phi}(x) = \int d\tilde{k}\,\left[\hat{a}(\mathbf{k})e^{-ikx} + \hat{a}^\dagger(\mathbf{k})e^{+ikx}\right],$$

show that

$$e^{i\lambda\cdot\hat{P}}\hat{\phi}(x)e^{-i\lambda\cdot\hat{P}} = \int d\tilde{k}\,\left[\hat{a}(\mathbf{k})e^{-ik(x+\lambda)} + \hat{a}^\dagger(\mathbf{k})e^{+ik(x+\lambda)}\right] = \hat{\phi}(x+\lambda).$$

Conclusion

$$\boxed{e^{i\lambda\cdot\hat{P}}\hat{\phi}(x)e^{-i\lambda\cdot\hat{P}} = \hat{\phi}(x+\lambda)},$$

so the argument of the field operator is changed by the generator of translations, as one would expect for a translation.

4.3 Complex KG Field

We now consider a complex KG field. The Lagrangian density is given by

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi = -\phi^*(\Box + m^2)\phi. \tag{4.52}$$

The discussion now runs parallel with the one we had for real KG fields. Therefore, we skip most derivations and discussions and only write down explicitly the most relevant results.

EoM: For the E-L EoM, we get

$$(\Box + m^2)\phi(x) = 0\,, \qquad (\Box + m^2)\phi^*(x) = 0\,. \tag{4.53}$$

Quantization: We impose the equal-time canonical commutation relations

$$[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta^{(d)}(\mathbf{x}-\mathbf{y}), \qquad [\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = 0, \tag{4.54}$$

and similarly for the conjugate.

Using $\hat{\pi}(\mathbf{x}) = \dot{\hat{\phi}}^\dagger(\mathbf{x})$, we can rewrite the first commutation relation above as

$$[\hat{\phi}(\mathbf{x}), \dot{\hat{\phi}}^\dagger(\mathbf{y})] = i\delta^{(d)}(\mathbf{x}-\mathbf{y}). \tag{4.55}$$

and similarly for the conjugate, where we have $\hat{\pi}^\dagger(\mathbf{x}) = \dot{\hat{\phi}}(\mathbf{x})$.

Solution to the EoM: The field operator $\hat{\phi}(x)$ can be expanded as

$$\hat{\phi}(x) = \int d\tilde{k}\left[\hat{a}(\mathbf{k})e^{-ikx} + \hat{b}^\dagger(\mathbf{k})e^{+ikx}\right], \tag{4.56}$$

and similarly

$$\hat{\phi}^\dagger(x) = \int d\tilde{k}\left[\hat{b}(\mathbf{k})e^{-ikx} + \hat{a}^\dagger(\mathbf{k})e^{+ikx}\right]. \tag{4.57}$$

Here we do not impose hermiticity. Therefore, $\hat{a}$ and $\hat{b}$ do not have to be equal (though they correspond to particles with the same mass, as they satisfy the same dispersion relation).

Commutation Relations: The canonical quantization imposes the following non-zero commutators

$$\begin{aligned}[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] &= 2\omega_k(2\pi)^d\delta^{(d)}(\mathbf{k}-\mathbf{k}'),\\ [\hat{b}(\mathbf{k}), \hat{b}^\dagger(\mathbf{k}')] &= 2\omega_k(2\pi)^d\delta^{(d)}(\mathbf{k}-\mathbf{k}').\end{aligned} \tag{4.58}$$

All other commutators vanish

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = [\hat{b}, \hat{b}] = [\hat{b}^\dagger, \hat{b}^\dagger] = [\hat{a}, \hat{b}] = \cdots = 0.$$

Two-particle Interpretation: This formalism introduces two operators: a and b, suggesting the presence of two kinds of particles with the same mass. The natural question is

How can we distinguish them?

The answer lies in the symmetries of the Lagrangian: the theory has a global $U(1)$ symmetry, which according to Noether's theorem implies the existence of a conserved charge.

We explore this symmetry and its associated charge below.

Global $U(1)$ Symmetry and Conserved Charge

The Lagrangian of the complex KG field is invariant under global $U(1)$ transformations (see also Exercise 7 in Sec. 1.3.5)

$$\phi(x) \to e^{i\alpha}\phi(x) \simeq \phi(x) + \delta\phi(x), \qquad \phi^*(x) \to e^{-i\alpha}\phi^*(x) \simeq \phi^*(x) - \delta\phi^*(x), \tag{4.59}$$

where the infinitesimal variation is

$$\delta\phi = i\alpha\phi(x), \qquad \delta\phi^* = -i\alpha\phi^*(x). \tag{4.60}$$

This transformation leaves the Lagrangian invariant:

$$\mathcal{L}' \equiv \mathcal{L}(\phi', \partial_\mu\phi') = \mathcal{L}(\phi, \partial_\mu\phi) \quad \Rightarrow \quad \partial_\mu\Omega^\mu = 0. \tag{4.61}$$

Noether's Current: Using Noether's theorem, we can derive the conserved current associated with this global $U(1)$ symmetry:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* = i\alpha\left[(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*\right] = -i\alpha\left(\phi^*\overleftrightarrow{\partial^\mu}\phi\right). \tag{4.62}$$

Noether's Charge: The conserved charge (integral over the time component of the current) is

$$Q = \int d^d\mathbf{x}\, j^0 = -iq\int d^d\mathbf{x}\left[\dot{\phi}^\dagger\phi - \dot{\phi}\phi^\dagger\right]. \tag{4.63}$$

Note that the normalization is arbitrary. We choose this one for convenience.

Exercise

Operator Form of the Charge: Using the mode expansion of the field operator show that

$$\hat{Q} = q\int d\tilde{k}\left[\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) - \hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k})\right]. \tag{4.64}$$

This charge distinguishes particles created by $\hat{a}^\dagger$ (charge $+q$) and those created by $\hat{b}^\dagger$ (charge $-q$).

Number Operator: We can also define a number operator that counts both types of particles regardless of charge

$$\hat{N} = \int d\tilde{k}\left[\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k})\right]. \tag{4.65}$$

Charge and Number Operator Eigenstates

We now define the one-particle states associated with the creation operators as

$$|\mathbf{k}\rangle_a \equiv \hat{a}^\dagger(\mathbf{k})|0\rangle, \qquad |\mathbf{k}\rangle_b \equiv \hat{b}^\dagger(\mathbf{k})|0\rangle. \tag{4.66}$$

Action of the Charge Operator: The charge operator acts on these states as

$$\hat{Q}\,|\mathbf{k}\rangle_a = q\int d\tilde{k}'\left[\hat{a}^\dagger(\mathbf{k}')\hat{a}(\mathbf{k}') - \hat{b}^\dagger(\mathbf{k}')\hat{b}(\mathbf{k}')\right]\hat{a}^\dagger(\mathbf{k})\,|0\rangle = +q\,|\mathbf{k}\rangle_a\,,$$
$$\hat{Q}\,|\mathbf{k}\rangle_b = -q\,|\mathbf{k}\rangle_b\,.$$

Action of the Number Operator

$$\hat{N}\,|\mathbf{k}\rangle_a = 1\,|\mathbf{k}\rangle_a\,,\qquad \hat{N}\,|\mathbf{k}\rangle_b = 1\,|\mathbf{k}\rangle_b\,. \tag{4.67}$$

Interpretation: The operator $\hat{Q}$ distinguishes between particles of

- Equal mass.
- Opposite charge.

Commutators and Microcausality

- One can show that the field commutators satisfy
$$[\hat{\phi}(x), \hat{\phi}^\dagger(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0.$$
This yields microcausality for complex KG fields.
- We also have
$$[\hat{\phi}(x), \hat{\phi}(y)] = 0 \quad \text{always.}$$

4.4 Wick's Theorem and Contractions

Wick's theorem relates time-ordered operators with normal-ordered operators plus Feynman propagators. This theorem will prove useful when we compute the S-matrix in perturbation theory.

We consider (relativistic) fields that are linear combinations of creation and annihilation operators: $\hat{\phi} = \hat{\phi}^{(+)} + \hat{\phi}^{(-)}$ with $\hat{\phi}^{(+)} \sim \hat{a}$ and $\hat{\phi}^{(-)} \sim \hat{a}^\dagger$ (or $\sim \hat{b}^\dagger$ if the field is complex). We will discuss both relativistic bosons and fermions.

Definition. Time ordering

We now give the general definition of the time ordering operation on n fields:

$$\begin{aligned} &T\{\hat{A}_1(x_1)\hat{A}_2(x_2)\cdots\hat{A}_n(x_n)\} \\ &= \sum_{\substack{\mathcal{P}erm \\ (\sigma\in\mathcal{P}_n)}} (-1)^{\mathcal{P}_\sigma}\theta(t_{\sigma_1}-t_{\sigma_2})\cdots\theta(t_{\sigma_{n-1}}-t_{\sigma_n})\hat{A}_{\sigma_1}(x_{\sigma_1})\cdots\hat{A}_{\sigma_n}(x_{\sigma_n})\,. \end{aligned} \tag{4.68}$$

$(-1)^{\mathcal{P}_\sigma}$ stands for the fact that a (-1) factor has to be introduced each time two fermion fields are permuted.

Definition. Normal ordering (general)

$$\hat{\phi} = \hat{\phi}^{(+)} + \hat{\phi}^{(-)}, \tag{4.69}$$

where $\hat{\phi}^{(+)} \equiv$ annihilation and $\hat{\phi}^{(-)} \equiv$ creation. Given fields $\hat{Q}, \hat{R}, \dots, \hat{W}$ made up of $\hat{\phi}^{(\pm)}$, then

$$: \hat{Q}^{(+)}\hat{R}^{(+)}\hat{S}^{(-)} \dots \hat{T}^{(+)} := (-1)^P \hat{S}^{(-)} \dots \hat{Q}^{(+)}\hat{R}^{(+)}\hat{T}^{(+)} \sim (\hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger)(\hat{a} \dots \hat{a}), \tag{4.70}$$

where P is the number of permutations (for fermions).

Notation. Contraction

We define the contraction of two fields as

$$\boxed{\overline{\hat{A}(x_1)\hat{B}(x_2)} \equiv \langle 0| T\left\{\hat{A}(x_1)\hat{B}(x_2)\right\}|0\rangle\ .} \tag{4.71}$$

Notation

$$: \hat{A}\hat{B}\hat{C}\hat{D}E, \dots, \hat{J}\hat{K}\hat{L}M \dots : = (-1)^P\ \overline{\hat{A}\,\hat{K}}\ \overline{\hat{B}\,\hat{C}}\ \overline{\hat{E}\,\hat{L}} : D\,F, \dots, J\,M, \dots : \tag{4.72}$$

where P is the number of permutations (for fermions) needed to reorder the terms in the left-hand side to match the order in the right-hand side.

Wick's theorem for two fields

Exercise

Consider the fields $\hat{A} = \hat{A}^{(+)} + \hat{A}^{(-)}$ and $\hat{B} = \hat{B}^{(+)} + \hat{B}^{(-)}$, where the (+) and (-) denote the terms proportional to the annihilation and creation operators, respectively. Prove that

$$\begin{aligned} T\left\{\hat{A}(x_1)\hat{B}(x_2)\right\} &= : \hat{A}(x_1)\hat{B}(x_2) : + \langle 0| T\left\{\hat{A}(x_1)\hat{B}(x_2)\right\}|0\rangle \\ &= : \hat{A}(x_1)\hat{B}(x_2) : + \overline{\hat{A}(x_1)\hat{B}}(x_2) \end{aligned} \tag{4.73}$$

for both bosons and fermions.

For three fields

$$\begin{aligned} T\left\{\hat{A}(x_1)\hat{B}(x_2)\hat{C}(x_3)\right\} &= : \hat{A}(x_1)\hat{B}(x_2)\hat{C}(x_3) : \\ &\quad + \overline{\hat{A}(x_1)\hat{B}}(x_2)\hat{C}(x_3) + A\ \overline{\hat{B}\ \hat{C}} + \overline{\hat{A}\,\hat{B}\,\hat{C}}. \end{aligned} \tag{4.74}$$

Wick's theorem (general)

all different contractions

$$T\left\{\hat{A}\hat{B}\hat{C}\hat{D}\ldots\hat{W}\hat{X}\hat{Y}\hat{Z}\right\} =: \hat{A}\hat{B}\hat{C}\hat{D}\ldots\hat{W}\hat{X}\hat{Y}\hat{Z}: \tag{4.75}$$

$$+: \overbracket{\hat{A}\hat{B}}\hat{C}\hat{D}\ldots\hat{W}\hat{X}\hat{Y}\hat{Z}: + \ldots + : \hat{A}\hat{B}\hat{C}\hat{D}\ldots\hat{W}\hat{X}\overbracket{\hat{Y}\hat{Z}}:$$

$$+\text{terms with two contractions} + \cdots$$

4.5 Exercises

1. Rewrite the generators of the Lorentz group of the real/complex Klein-Gordon field obtained in exercise 7 of Sec. 1.3.5 in terms of the creation and annihilation operators.

2. Determine
$$\hat{U}_0(\Lambda)\hat{\phi}^{(+)}(x)\hat{U}_0^{-1}(\Lambda)$$
if
$$\hat{U}_0(\Lambda)\hat{a}(\mathbf{k})\hat{U}_0^{-1}(\Lambda) = \hat{a}(\Lambda\mathbf{k})\,.$$

3. Consider the operator:
$$\hat{K}^i \equiv -i\int d\tilde{k}\hat{a}^\dagger(\mathbf{k})w_k\left(\frac{\partial}{\partial k^i}\hat{a}(\mathbf{k})\right)$$
and compute ($\hat{n}^2 = 1$)
$$e^{i\lambda\hat{n}\cdot\hat{\mathbf{K}}}\hat{a}(\mathbf{q})e^{-i\lambda\hat{n}\cdot\hat{\mathbf{K}}} \qquad e^{i\lambda\hat{n}\cdot\hat{\mathbf{K}}}\hat{a}^\dagger(\mathbf{q})e^{-i\lambda\hat{n}\cdot\hat{\mathbf{K}}}$$
to order λ. If possible interpret the result.

4. Compute
$$[\hat{\phi}^3(x), \hat{\phi}^3(y)]$$
for space-like distances.

5. **Number operator** in the relativistic theory.
a) Using Eq. (3.33) for a spin-zero particle show that
$$\hat{N} = i\int_{R^3} d^d\mathbf{x}\left(\hat{\phi}^{(-)}(x)\left(\frac{\partial}{\partial t}\hat{\phi}^{(+)}(x)\right) - \left(\frac{\partial}{\partial t}\hat{\phi}^{(-)}(x)\right)\hat{\phi}^{(+)}(x)\right)$$
b) If we define
$$\hat{N}_V(t) = i\int_{V} d^d\mathbf{x}\left(\hat{\phi}^{(-)}(x)\left(\frac{\partial}{\partial t}\hat{\phi}^{(+)}(x)\right) - \left(\frac{\partial}{\partial t}\hat{\phi}^{(-)}(x)\right)\hat{\phi}^{(+)}(x)\right)$$
compute and interpret $[\hat{N}_V(t), \hat{\phi}^{(-)}(t,\mathbf{x})]$ and $[\hat{N}_V(t), \hat{N}_{V'}(t)]$.

6. **Charge operator** in the relativistic theory.
   a) Using Eq. (4.64) show that

$$\hat{Q} = -iq \int_{R^3} d^d\mathbf{x} \left(\left(\frac{\partial}{\partial t} \hat{\phi}^\dagger(x) \right) \hat{\phi}(x) - \left(\frac{\partial}{\partial t} \hat{\phi}(x) \right) \hat{\phi}^\dagger(x) \right)$$

   b) If we define

$$\hat{Q}_V(t) = -iq \int_V d^d\mathbf{x} \left(\left(\frac{\partial}{\partial t} \hat{\phi}^\dagger(x) \right) \hat{\phi}(x) - \left(\frac{\partial}{\partial t} \hat{\phi}(x) \right) \hat{\phi}^\dagger(x) \right)$$

   compute and interpret $[\hat{Q}_V(t), \hat{\phi}^\dagger(\mathbf{x}, t)]$, $[\hat{Q}_V(t), \hat{Q}_{V'}(t)]$, and $[\hat{Q}_V(t), \hat{N}]$.

7. Consider the operator $\hat{\mathcal{P}} \equiv \hat{\mathcal{P}}_1 \hat{\mathcal{P}}_2$ where ($\eta_p = \pm 1$)

$$\hat{\mathcal{P}}_1 = \exp\left[-i\frac{\pi}{2} \int d\tilde{k} \hat{a}^\dagger(\mathbf{k}) \hat{a}(\mathbf{k}) \right] \quad \text{and} \quad \hat{\mathcal{P}}_2 = \exp\left[i\frac{\pi}{2}\eta_p \int d\tilde{k} \hat{a}^\dagger(\mathbf{k}) \hat{a}(-\mathbf{k}) \right]$$

   and compute

$$\hat{\mathcal{P}}^2, \quad \hat{\mathcal{P}} \hat{a}(\mathbf{k}) \hat{\mathcal{P}}^\dagger, \quad \hat{\mathcal{P}} \hat{a}^\dagger(\mathbf{k}) \hat{\mathcal{P}}^\dagger, \quad \hat{\mathcal{P}} \hat{\phi}(x) \hat{\mathcal{P}}^\dagger, \quad \hat{\mathcal{P}} \hat{\mathbf{P}} \hat{\mathcal{P}}^\dagger, \quad \hat{\mathcal{P}} \hat{H} \hat{\mathcal{P}}^\dagger .$$

8. Consider the operator

$$\hat{\mathcal{C}} = \exp\left[-i\frac{\pi}{2} \int d\tilde{k} \left(\hat{a}^\dagger(\mathbf{k}) \hat{a}(\mathbf{k}) - \hat{a}^\dagger(\mathbf{k}) \hat{b}(\mathbf{k}) + \hat{b}^\dagger(\mathbf{k}) \hat{b}(\mathbf{k}) - \hat{b}^\dagger(\mathbf{k}) \hat{a}(\mathbf{k}) \right) \right]$$

   and compute

$$\hat{\mathcal{C}}^2, \quad \hat{\mathcal{C}} \hat{a}(\mathbf{k}) \hat{\mathcal{C}}^\dagger, \quad \hat{\mathcal{C}} \hat{b}(\mathbf{k}) \hat{\mathcal{C}}^\dagger, \quad \hat{\mathcal{C}} \hat{a}^\dagger(\mathbf{k}) \hat{\mathcal{C}}^\dagger, \quad \hat{\mathcal{C}} \hat{b}^\dagger(\mathbf{k}) \hat{\mathcal{C}}^\dagger,$$

$$\hat{\mathcal{C}} \hat{\phi}(x) \hat{\mathcal{C}}^\dagger, \quad \hat{\mathcal{C}} \hat{\mathbf{P}} \hat{\mathcal{C}}^\dagger, \quad \hat{\mathcal{C}} \hat{H} \hat{\mathcal{C}}^\dagger .$$

5 S-matrix (interactions)

As we have already emphasized in this book, QFT is nothing but quantum mechanics. The description of any physical process is made via the evaluation of the appropriate matrix element. There are three pictures that are used to compute them. We recall them and discuss their advantages and disadvantages for a QFT.

We begin by working in the Schrödinger picture:

- Initial state: $|\alpha, t'\rangle_S$
- Final state: $|\beta, t\rangle_S$

We are interested in the following matrix element[1]

$$ {}_S\langle\beta, t|\hat{U}(t,t')|\alpha, t'\rangle_S, \qquad \hat{U}(t,t') = e^{-i\hat{H}(t-t')}\,, \tag{5.1}$$

This matrix element describes the evolution of the state $|\alpha, t'\rangle$ with time and the projection onto the state $|\beta, t\rangle$. From this we can compute the measurable quantity:

$$ \left| {}_S\langle\beta, t|\hat{U}(t,t')|\alpha, t'\rangle_S \right|^2 \,, \tag{5.2}$$

which is the transition probability. We are particularly interested in computing this in the limit

$$ t \to +\infty, \quad t' \to -\infty\,. $$

We now want to discuss what these quantities look like in the Schrödinger, Heisenberg and Interaction pictures.

Observations:

- **Schrödinger**: More intuitive (our starting point and the starting point in standard quantum mechanics courses). We tend to think of the system (us and the universe) as a state (or a density matrix) in a Hilbert space.

[1]The time evolution operator $\hat{U}(t,t')$ should not be confused with the general unitary representation of the Poincare group $\hat{U}(\Lambda, a)$. On the other hand, it should also be noted that $\hat{U}(t,0) = \hat{U}(\mathbb{I}, -t)$.

- **Heisenberg**: States do not evolve in time; time evolution is encoded in the operators. This is more natural for QFTs where time and space are treated on an equal footing: $x = (t, \mathbf{x})$. This is particularly relevant in relativistic theories where we want to keep Lorentz invariance explicit.
- **Interaction picture**: The interaction picture is a mixture between the Schrödinger and Heisenberg pictures. It corresponds to the Heisenberg picture if the interaction term is zero.
  - Natural in perturbation theory (when there is a small interaction parameter).
  - Also appears when dealing with *asymptotic states* (as we will see).

We will **set all pictures to be equal at a reference time** t_0 **(**$= 0$**)**.

Our Primary Definition of the S-matrix

We define the S-matrix element in the Schrödinger picture as

$$\boxed{S_{\beta\alpha} \equiv \lim_{\substack{t \to +\infty \\ t' \to -\infty}} {}_S\langle \beta, t | \hat{U}(t, t') | \alpha, t' \rangle_S \,.} \tag{5.3}$$

Heisenberg Picture

How does Eq. (5.3) look like in the Heisenberg picture? At $t = t' = 0$, we have

$$|\alpha/\beta, t' = 0\rangle_S = |\alpha/\beta, t' = 0\rangle_H = |\alpha/\beta\rangle_H \,. \tag{5.4}$$

These states are time-independent. They are often called "in" for α and "out" for β (we discuss why they are called this way later). At different times, the equalities go as follows:

$$\begin{aligned} |\alpha\rangle_H &= \hat{U}^\dagger(t', 0)\, |\alpha, t'\rangle_S = e^{i\hat{H}t'}\, |\alpha, t'\rangle_S \ , \\ |\beta\rangle_H &= \hat{U}^\dagger(t, 0)\, |\beta, t\rangle_S = e^{i\hat{H}t}\, |\beta, t\rangle_S \ . \end{aligned}$$

In terms of Heisenberg states, the S-matrix element reads[2]

$$\boxed{S_{\beta\alpha} = {}_H\langle \beta | \alpha \rangle_H \equiv {}_H\langle \beta, \text{out} | \alpha, \text{in} \rangle_H \,.} \tag{5.5}$$

[2] This is the standard definition for S-matrix one can find in [3, 4, 10] for instance. However, the definition in [7] is different.

Interaction Picture

In the interaction picture, the Hamiltonian is split into two terms:

$$\hat{H} = \hat{H}_0 + \hat{H}_I \quad \text{where} \quad \hat{H}_0 \sim \hat{a}^\dagger \hat{a},\ H_I \sim \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} + \dots .$$

We take $\hat{H}_0$ to be quadratic in the creation/annihilation operators and $\hat{H}_I$ to be a polynomial of degree greater than or equal to three.

The interaction picture incorporates the time evolution associated with $\hat{H}_0$ into the fields. There is still a left-over time dependence on the states because of $\hat{H}_I$.

We now want to understand how the time evolution in the interaction picture relates to the time evolution in Schrödinger's picture. We start with

$$|\psi(t)\rangle_I \equiv \hat{U}_0^\dagger(t, t_0)\, |\psi(t)\rangle_S \ , \tag{5.6}$$

where

$$\hat{U}_0(t, t_0) = e^{-i(t-t_0)\hat{H}_0} \, . \tag{5.7}$$

On the other hand we have that

$$|\psi(t)\rangle_S = \hat{U}(t, t')\, |\psi(t')\rangle_S \quad \Rightarrow \quad |\psi(t)\rangle_I = \hat{U}_0^\dagger(t, t_0)\hat{U}(t, t')\, |\psi(t')\rangle_S \, . \tag{5.8}$$

Rewriting $|\psi(t')\rangle_S$ back into the interaction picture

$$|\psi(t')\rangle_S = \hat{U}_0(t', t_0)\, |\psi(t')\rangle_I \ ,$$

we obtain

$$|\psi(t)\rangle_I = \hat{U}_0^\dagger(t, t_0)\hat{U}(t, t')\hat{U}_0(t', t_0)\, |\psi(t')\rangle_I \, .$$

If we fix $t_0 = 0$

$$\Rightarrow \boxed{|\psi(t)\rangle_I = e^{it\hat{H}_0} e^{-i(t-t')\hat{H}} e^{-it'\hat{H}_0}\, |\psi(t')\rangle_I \, .}$$

This motivates the following definition of the time evolution operator in the interaction picture:

$$\boxed{\hat{U}_I(t, t') = e^{it\hat{H}_0} e^{-i(t-t')\hat{H}} e^{-it'\hat{H}_0} \, .} \tag{5.9}$$

It satisfies

$$|\psi(t)\rangle_I = \hat{U}_I(t, t')\, |\psi(t')\rangle_I \ ,$$

and the S-matrix reads

$$\boxed{S_{\beta\alpha} = \lim_{\substack{t \to +\infty \\ t' \to -\infty}} {}_I\langle \beta, t | \hat{U}_I(t, t') | \alpha, t' \rangle_I \, .}$$

Asymptotic states vs free states vs the S-matrix

By construction, the three pictures are nothing but rewriting in different ways the same matrix element:

$$ {}_S\langle\psi(t)|\hat{O}^S|\psi(t)\rangle_S = {}_H\langle\psi|\hat{O}^H(t)|\psi\rangle_H = {}_I\langle\psi(t)|\hat{O}^I(t)|\psi(t)\rangle_I \,. $$

The S-matrix elements have been defined in the Schrödinger picture taking the $t \to +\infty$, $t' \to -\infty$ limit, but it looks like we could have connected with the Schrödinger picture at any time. This links with the following question.

Question: Does the S-matrix element depend on t and t'? It doesn't seem so in the Heisenberg picture.

Answer: The answer is **no**. There is **no time dependence**. In the way we define them, the S-matrix elements yield the same result regardless of the values of t and t' in the Schrödinger picture.
What is the reason for that? This is due to the meaning we give to $|\alpha, t'\rangle_S$. In other words, how do we interpret/define the label "α"? The point is that these states are characterized at some <u>fixed</u> reference time. In our case, these reference times are taken to be

$$ t \to +\infty, \quad t' \to -\infty \,. $$

that is, the "in" and "out" states. At those times, we take the states to be direct products of one-particle states, i.e. direct product of

unitary irreducible representations of the Poincaré group.

The states $|\alpha, t\rangle_S \,, |\beta, t'\rangle_S$ should be interpreted as describing how these direct products of free particles evolve up to t and t'. The states $|\alpha\rangle_H \,, |\beta\rangle_H$ in the Heisenberg picture should then be interpreted as describing how these direct products of free particles evolve up to $t = t' = 0$. This yields the link between the free particles we have characterized in Chapter 3 and the asymptotic states we measure in detectors. This will be the key point below to link with the interaction picture.

To make this consistent, it is necessary that in the $t \to \pm\infty$ limit, the states can be characterized by a set of numbers which are time independent. It is here where the main assumption comes in, which we quantify below.

We first have the following equality

$$ |\alpha, \text{in}\rangle_H \underset{\uparrow}{=} \lim_{t\to-\infty} \hat{U}^\dagger(t,0)\, |\alpha, t\rangle_S \underset{\uparrow}{=} \lim_{t\to-\infty} \hat{U}^\dagger(t,0)\hat{U}_0(t,0)\, |\alpha, t\rangle_I \,. $$

These equalities hold true for any t.

Assumption:[3]

$$|\alpha, t\rangle_I \to |\alpha\rangle\,, \quad \text{as} \quad t \to -\infty \quad \text{(time-independent)}$$

- We *switch off all interactions* at $t \to \pm\infty$. Therefore, $|\alpha\rangle$ is an eigenstate of H_0. In the interaction picture, any time dependence of the states is due to the interaction, which vanishes in the $t \to \pm\infty$ limit.
- $|\alpha\rangle = \hat{a}^\dagger(\mathbf{p}_1)\hat{a}^\dagger(\mathbf{p}_2)\cdots\hat{a}^\dagger(\mathbf{p}_n)\,|0\rangle\,, \qquad |\beta\rangle = \hat{a}^\dagger(\mathbf{p}'_1)\hat{a}^\dagger(\mathbf{p}'_2)\cdots\hat{a}^\dagger(\mathbf{p}'_m)\,|0\rangle\,.$

$$|\alpha, \text{in}\rangle = \lim_{t\to-\infty} e^{it\hat{H}_0}e^{-it\hat{H}}\,|\alpha\rangle\,; \qquad |\beta, \text{out}\rangle = \lim_{t\to\infty} e^{it\hat{H}}e^{-it\hat{H}_0}\,|\beta\rangle.$$

We finally obtain

$$\boxed{S_{\beta\alpha} = \langle\beta|\hat{S}|\alpha\rangle, \quad \text{where} \quad \hat{S} = \hat{U}_I(\infty, -\infty)\,.} \tag{5.10}$$

The asymptotic states are completely characterized by the unitary irreducible representations of the symmetry group of the theory. What is left now is the determination of $\hat{S}$. One can first realize that the S-operator is unitary:

$$\hat{S}^\dagger\hat{S} = \hat{S}\hat{S}^\dagger = I \qquad \hat{S}^\dagger = \hat{S}^{-1}\,. \tag{5.11}$$

For a complete characterization of $\hat{S}$, it is convenient to write down the time-evolution differential equation of the S-operator, which we discuss next.

5.1 Time Evolution in the Interaction Picture

Time evolution in the Three Pictures

Schrödinger
The time evolution of the states in the Schrödinger picture goes as follows

$$i\frac{d}{dt}\,|\psi(t)\rangle_S = \hat{H}\,|\psi(t)\rangle_S\,, \quad \text{(We take } \hat{H} \text{ to be time independent)}$$

$$\Rightarrow |\psi(t)\rangle_S = \hat{U}(t, t_0)\,|\psi(t_0)\rangle_S\,, \quad \hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)}\,,$$

whereas we take the operators to be time independent: $\hat{O}_S$.

Heisenberg

$$|\psi\rangle_H \equiv \hat{U}^\dagger(t, t_0)\,|\psi(t)\rangle_S \quad \Rightarrow |\psi\rangle_H \text{ does not depend on } t\,.$$

[3] For simplicity, we will always work with a ket/bra notation ($\langle / \rangle$), even for the asymptotic states. In [3, 4], the states we denote for $|\alpha\rangle$, the (time-independent) asymptotic states, are written as $|\alpha)$.

Schrödinger and Heisenberg pictures are equal at t_0. We omit the dependence on t_0 below. Then,

$$\hat{O}^H(t) = \hat{U}^\dagger(t)\hat{O}^S\hat{U}(t), \quad i\frac{d}{dt}\hat{O}^H(t) = [\hat{O}^H(t), \hat{H}]\,.$$

These are the time-evolution equations if there is no explicit time dependence on O_S.

Interaction
The time evolution of the states has been discussed before. For operators, we have

$$\hat{O}^I(t) = \hat{U}_0^T(t,t_0)\hat{O}^S\hat{U}_0(t,t_0)\,,$$

or, in its differential form,

$$i\frac{d}{dt}\hat{O}^I(t) = [\hat{O}^I(t), \hat{H}_0]\,.$$

These are the EoM we had for the KG field in Chapter 4.1. Alternatively, they can also be interpreted as time translations:

$$e^{ix^0\hat{P}^0}\underbrace{\hat{\phi}(0,\mathbf{x})}_{\equiv\hat{\phi}_S(\mathbf{x})}e^{-ix^0\hat{P}^0} = \hat{\phi}_H(x) = \hat{\phi}_I(x)\,,$$

where the last equality holds for free fields ($\hat{P}^0 = \hat{H}_0$). Note that states and operators are set to be equal at a fixed time $t_0 = 0$.

We now want to give an explicit expression for the time evolution operator $\hat{U}_I(t,t')$. To do this, we compute its derivative

$$i\frac{d}{dt}\hat{U}_I(t,t') = i\frac{d}{dt}\left[e^{it\hat{H}_0}e^{-i(t-t')\hat{H}}e^{-it'\hat{H}_0}\right]\,.$$

After applying the chain rule and using commutator identities, we find

$$i\frac{d}{dt}\hat{U}_I(t,t') = \hat{H}_I^I(t)\hat{U}_I(t,t'),$$

where

$$\hat{H}_I^I(t) = e^{it\hat{H}_0}\hat{H}_I e^{-it\hat{H}_0} \quad \text{(interaction Hamiltonian in the interaction picture).}$$

Integral Form. We can write this equation as an integral equation:

$$\hat{U}_I(t,t') = \mathbb{I} - i\int_{t'}^{t} d\tau\, \hat{H}_I^I(\tau)\hat{U}_I(\tau,t')\,.$$

We can now solve this equation iteratively (Dyson series)

$$\begin{aligned}\hat{U}_I(t,t') &= \mathbb{I} - i\int_{t'}^{t} d\tau\, \hat{H}_I^I(\tau) + (-i)^2\int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2\, \hat{H}_I^I(\tau_1)\hat{H}_I^I(\tau_2) + \cdots \\ &= \sum_{n=0}^{\infty}(-i)^n \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \cdots \int_{t'}^{\tau_{n-1}} d\tau_n\, \hat{H}_I^I(\tau_1)\cdots \hat{H}_I^I(\tau_n)\,.\end{aligned}$$

The nested integrals impose the ordering

$$\tau_1 > \tau_2 > \cdots > \tau_n \quad \Rightarrow \quad \textbf{time-ordering!}$$

Dyson Series and Time-Ordering Notation

We can express the Dyson series using the **time-ordering operator** T. This allows us to write

$$\hat{U}_I(t,t') = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} d\tau_1 \cdots \int_{t'}^{t} d\tau_n\, T\left\{\hat{H}_I^I(\tau_1)\cdots \hat{H}_I^I(\tau_n)\right\}\,, \tag{5.12}$$

which, in compact form, becomes

$$\boxed{\hat{U}_I(t,t') = T\exp\left(-i\int_{t'}^{t} d\tau\, \hat{H}_I^I(\tau)\right)\,,} \tag{5.13}$$

where the operation T (an operator acting on the set of operators that act on the Hilbert space) ensures the product is time-ordered:

$$T\left\{\hat{H}_I^I(\tau_1)\hat{H}_I^I(\tau_2)\right\} = \begin{cases}\hat{H}_I^I(\tau_1)\hat{H}_I^I(\tau_2), & \text{if } \tau_1 > \tau_2,\\ \hat{H}_I^I(\tau_2)\hat{H}_I^I(\tau_1), & \text{if } \tau_2 > \tau_1.\end{cases}$$

Summary: The full evolution of the system in the interaction picture is governed by

$$|\psi(t)\rangle_I = \hat{U}_I(t,t')\,|\psi(t')\rangle_I\,, \qquad \hat{U}_I(t,t') = T\exp\left(-i\int_{t'}^{t} d\tau\, \hat{H}_I^I(\tau)\right)$$

and

$$\boxed{\boxed{\hat{S} = T\left\{e^{-i\int_{-\infty}^{+\infty} d\tau\, \hat{H}_I^I(\tau)}\right\}.}} \tag{5.14}$$

If the (interaction) Hamiltonian can be written in terms of an (interaction) Hamiltonian density:

$$\hat{H}_I^I(t) = \int d^d\mathbf{x}\,\hat{\mathcal{H}}_I^I(x), \tag{5.15}$$

the S-operator can be written as:

$$\boxed{\boxed{\hat{S} = T\left\{e^{-i\int d^D x\,\hat{\mathcal{H}}_I^I(x)}\right\}.}} \tag{5.16}$$

5.2 Motivation for Causal (free) Fields

Asymptotic states are naturally characterized as direct products of one-particle unitary irreducible representations of the Poincaré group. The quantum numbers of these depend on the momentum. Furthermore, the S-operator is a polynomial in powers of creation and annihilation operators of these states. Therefore, one may wonder why we should bother constructing fields in position space. As emphasized in [4], the key reason is that the momentum dependence of the unitary representations of the Poincaré transformations in momentum space makes the construction of Poincaré-invariant interaction terms very complicated. This problem can be overcome to a large extent by using fields with suitable transformation properties under Poincaré transformations. This is the main reason we construct fields in position space.

Let us first recall the transformation properties of one-particle states (see Eq. (3.15)):

$$\hat{U}_0(\Lambda, a)\hat{a}_\sigma^\dagger(p)\left|0\right\rangle = e^{i(\Lambda p)\cdot a}\sum_{\sigma'} T_{\sigma'\sigma}^{(j)}(R(\Lambda, p))\hat{a}_{\sigma'}^\dagger(\Lambda p)\left|0\right\rangle, \tag{5.17}$$

where $\hat{U}_0(\Lambda, a)$ is a unitary representation of the Poincaré group for free particles; $T^{(j)}$ is a unitary representation of the little group for particles with spin j, and $R(\Lambda, p)$ is the Wigner rotation. For massless particles it also has this form but the unitary representation T is a pure phase.

Regardless of whether we consider massive or massless particles, we can see that $T_{\sigma'\sigma}^{(j)}$ depends explicitly on the momentum p. To avoid this problem, we construct fields in position space, whose transformations under Poincaré transformations are x-independent and homogeneous:

$$\hat{U}_0(\Lambda, a)\hat{\Psi}_l^{(+)}(x)\hat{U}_0^\dagger(\Lambda, a) = \sum_{l'} T_{l\,l'}(\Lambda^{-1})\hat{\Psi}_{l'}^{(+)}(\Lambda x + a). \tag{5.18}$$

These transformation laws formally hold both for massive and massless particles. We emphasize that now $T_{l\,l'}$ is momentum independent. We do not state it explicitly but T will depend on the representation. Moreover, these representations are typically finite-dimensional. The price to be paid is that these representations are now non-unitary. Also note that Eq. (5.18) yields the analogous symmetry transformations to those obtained in quantum mechanics courses. This is why we have $T_{l\,l'}(\Lambda^{-1})$, with Λ^{-1}, the inverse element of the Lorentz transformation. If we had a vector it would transform in the inverse way, just as would occur in quantum mechanics if we perform a rotation on a vector operator such as $\hat{\mathbf{X}} \to \hat{U}(R)\hat{\mathbf{X}}\hat{U}^{\dagger}(R) = R^{-1}\hat{\mathbf{X}}$. This should not be confused with how the argument of the field transforms. For these, the transformation above is the opposite of the convention we took in Sec. 1.4.3 for scalar fields. If restricted to rotations, that convention corresponds to the transformations one would have for a wave function in quantum mechanics: $\langle \mathbf{x}|\hat{U}(R)|\phi\rangle = \langle R^{-1}\mathbf{x}|\phi\rangle$.

The general expression for these causal fields can be written as

$$\hat{\Psi}_l^{(+)}(x) = \sum_{\sigma,n} \int d\tilde{p}\; u_l(\mathbf{p},\sigma,n) e^{-ipx}\; \hat{a}_{\sigma,n}(p), \tag{5.19}$$

where n stands for internal quantum numbers, in case the particle has any. Eq. (5.18) gives some constraints on the form of $u_l(\mathbf{p},\sigma,n)$. For the case of the scalar particle, the solution is trivial, and we have already obtained it in previous chapters:

$$\hat{\phi}^{(+)}(x) = \int d\tilde{k}\; \hat{a}(\mathbf{k})\; e^{-ikx}. \tag{5.20}$$

For a general spin, this problem may have no solution. Indeed, this is what happens with photons (see [4]). We disregard this problem in this book, and still define photon fields in position space following Eq. (5.19). These photon fields, however, will not satisfy Eq. (5.18) (see Chapter 7).

One may wonder why the causal fields should be linear combinations of annihilation (or creation) operators. The answer is that this is not compulsory at all. More complex dependencies on creation and annihilation operators are indeed possible. Actually, they could even be useful in particular problems, such as when dealing with photons, gluons or gravitons. Examples include Wilson lines and the like, or the metric in the case of gravitons. In any case, these nonlinear operators can typically be understood as functions of the linear operators considered in this book. Therefore, we will not study such constructions here and will stick to the standard linear dependence.

5.3 How to Implement Lorentz Invariance in the S-Matrix

We demand the S-operator in relativistically invariant QFTs to fulfill the following condition:

$$\boxed{\hat{U}_0(\Lambda, a)\, \hat{S}\, \hat{U}_0^\dagger(\Lambda, a) = \hat{S} \quad \forall\ (\Lambda, a)\,.} \tag{5.21}$$

This condition guarantees that the S-matrix is (unitarily) covariant under Poincaré transformations.

The next point is to determine the conditions that Eq. (5.21) imposes on the form of the interaction Hamiltonian. Here, we follow the discussion in [4, 3], and focus on interaction Hamiltonians that can be expressed in terms of a Hamiltonian density (see Eq. (5.15)). We now present sufficient conditions on $\hat{\mathcal{H}}_I^I(x)$ to ensure that the S-operator is a scalar under Poincaré transformations. They are two, which we now list.

We demand that $\hat{\mathcal{H}}_I^I(x)$ transforms as a scalar under Lorentz transformations:

$$\boxed{1) \qquad \hat{U}_0(\Lambda, a)\hat{\mathcal{H}}_I^I(x)\hat{U}_0^\dagger(\Lambda, a) = \hat{\mathcal{H}}_I^I(\Lambda x + a).} \tag{5.22}$$

Then the full integrand transforms as

$$\begin{aligned}
&\hat{U}_0(\Lambda, a)\left(\int d^D x\, \hat{\mathcal{H}}_I^I(x)\right)\hat{U}_0^\dagger(\Lambda, a) = \int d^D x\, \hat{\mathcal{H}}_I^I(\Lambda x + a) \\
&\overset{\Lambda x + a = x'}{=} \int d^D x\, \hat{\mathcal{H}}_I^I(x') = \int d^D x\, \hat{\mathcal{H}}_I^I(x),
\end{aligned} \tag{5.23}$$

since $|\det \Lambda| = 1$ is the modulus of the Jacobian of a Poincaré transformation.

Condition 1) is not enough. We also need the **time ordering** to be preserved under Lorentz transformations (remember that we restrict ourselves to proper orthochronous Lorentz transformations). The integrals over space-time coordinates force us to consider general $x^\mu \in \mathbb{R}^4$ vectors. If $x^2 \geq 0$ (time-like or light-like vectors), the time ordering is preserved. Nevertheless, this is not so for space-like vectors ($x^2 < 0$), since Lorentz transformations can change the sign of x^0. This problem is overcome by demanding

$$\boxed{2) \quad [\hat{\mathcal{H}}_I^I(x), \hat{\mathcal{H}}_I^I(x')] = 0 \quad \text{for} \quad (x - x')^2 < 0.} \tag{5.24}$$

For theories with scalar particles, this condition is guaranteed if $\hat{\mathcal{H}}_I^I(x)$ is written as a polynomial of $\hat{\phi}$, or its derivatives, because they fulfill microcausality:

$$[\hat{\phi}(x), \hat{\phi}(x')] = 0 \quad \text{if } (x - x')^2 < 0. \tag{5.25}$$

This is why we cannot work with $\phi^{(\pm)}(x)$ separately.

Conclusion: The building blocks of interaction terms must be **specific linear combinations** of creation and annihilation operators

$$\hat{H}_I \sim \int dk_i \dots a^\dagger a^\dagger \dots aa \tag{5.26}$$

fulfilling conditions 1) and 2).

We have provided sufficient conditions for ensuring Poincaré invariance when the interaction Hamiltonian is written in terms of a (local) Hamiltonian density. For theories with only scalar particles this is enough. Nevertheless, there are more general possibilities. These happen, for instance, if one has massless spin-one particles. Even though we will consider a relativistic theory of scalars and massless spin-one particles later, we will not write down explicitly the interaction Hamiltonian in this case. For a more thorough discussion, we refer the reader to [4].

5.4 Particles with Internal Quantum Numbers

We define the operator

$$\hat{Q} = \int d\tilde{k} \sum_n n\, \hat{a}^\dagger(\mathbf{k}, n)\hat{a}(\mathbf{k}, n). \tag{5.27}$$

This operator counts identical particles weighted by the internal quantum number n. For example, particles with the same mass but different charge.

For simplicity, we restrict the discussion to only scalar particles.

Exercise: Compute the following commutators

$$\begin{aligned}
[\hat{Q}, \hat{a}(\mathbf{k}, n)] &= \int d\tilde{k}' \sum_{n'} n' [\hat{a}^\dagger(\mathbf{k}', n')\hat{a}(\mathbf{k}', n'), \hat{a}(\mathbf{k}, n)] \\
&= \int d\tilde{k}' \sum_{n'} n' [\hat{a}^\dagger(\mathbf{k}', n'), \hat{a}(\mathbf{k}, n)]\hat{a}(\mathbf{k}', n') = -n\, \hat{a}(\mathbf{k}, n), \\
[\hat{Q}, \hat{a}^\dagger(\mathbf{k}, n)] &= \int d\tilde{k}' \sum_{n'} n' [\hat{a}^\dagger(\mathbf{k}', n')\hat{a}(\mathbf{k}', n'), \hat{a}^\dagger(\mathbf{k}, n)] \\
&= \int d\tilde{k}' \sum_{n'} n' \hat{a}^\dagger(\mathbf{k}', n')[\hat{a}(\mathbf{k}', n'), \hat{a}^\dagger(\mathbf{k}, n)] = +n\, \hat{a}^\dagger(\mathbf{k}, n).
\end{aligned}$$

From this, we find

$$\boxed{\hat{Q}\hat{a}^\dagger(\mathbf{k}, n) |0\rangle = n\, \hat{a}^\dagger(\mathbf{k}, n) |0\rangle\,.} \tag{5.28}$$

Causal Fields for $n = \pm|q|$

We now restrict to the particular case in which n can take two values: $n = \pm|q|$. Then, we can construct four causal fields

$$\begin{aligned} \hat{\phi}^{(+)}_{|q|} &\equiv \hat{\phi}^{(+)} \sim \hat{a}_{|q|}, \qquad & \hat{\phi}^{(-)}_{|q|} &\equiv \hat{\phi}^{(-)} \sim \hat{a}^{\dagger}_{|q|}, \\ \hat{\phi}^{(+)}_{-|q|} &\equiv \hat{\phi}^{(+)}_{c} \sim \hat{a}_{-|q|}, \qquad & \hat{\phi}^{(-)}_{-|q|} &\equiv \hat{\phi}^{(-)}_{c} \sim \hat{a}^{\dagger}_{-|q|}. \end{aligned}$$

Microcausality Condition: The interaction term must be a polynomial in the causal fields of the form

$$\hat{\phi}^{(+)}_{A} + \hat{\phi}^{(-)}_{B}\,.$$

In principle, any combination (A,B) generates a causal field that fulfills the microcausality condition (5.25). If we do not have any internal quantum number we end up with the real KG field

$$\hat{\phi} \sim \hat{a} + \hat{a}^{\dagger}\,.$$

Question: *What happens for particles with internal quantum numbers* $\pm|q|$*?*

We already have that

$$[\hat{Q}, \hat{\phi}^{(\pm)}_{|q|}] = \mp|q|\,\hat{\phi}^{(\pm)}_{|q|}, \qquad [\hat{Q}, \hat{\phi}^{(\pm)}_{-|q|}] = \pm|q|\,\hat{\phi}^{(\pm)}_{-|q|}.$$

To fix which particular combination of $\hat{\phi}^{(+)}_{A} + \hat{\phi}^{(-)}_{B}$ appears in the interaction term, we have to impose an extra condition. We demand that

$$[\hat{Q}, \hat{H}] = 0\,. \tag{5.29}$$

The free Hamiltonian

$$\hat{H}^{(0)} = \int d\tilde{k}\,\omega_k \sum_{\sigma(\textit{if they have spin})} \left(\hat{a}^{\dagger}_{\sigma}(\mathbf{k},|q|)\hat{a}_{\sigma}(\mathbf{k},|q|) + \hat{a}^{\dagger}_{\sigma}(\mathbf{k},-|q|)\hat{a}_{\sigma}(\mathbf{k},-|q|)\right),$$

already satisfies that $[\hat{Q}, \hat{H}^{(0)}] = 0$. Therefore, we have the following condition

$$\boxed{[\hat{Q}, \hat{H}_I] = 0\,.} \tag{5.30}$$

If we define

$$\hat{\phi} \equiv \hat{\phi}^{(+)} + \hat{\phi}^{(-)}_{c}, \tag{5.31}$$

this field satisfies the requirement of microcausality and transforms homogeneously under Q

$$\boxed{[\hat{Q}, \hat{\phi}] = -|q|\phi\,,} \quad \boxed{[\hat{Q}, \hat{\phi}^{\dagger}] = |q|\,\hat{\phi}^{\dagger}\,,}$$

where $\hat{\phi}^\dagger = \hat{\phi}_c^{(+)} + \hat{\phi}^{(-)}$. Therefore, any interaction Hamiltonian of the form

$$\boxed{H_I \sim (\hat{\phi}^\dagger)^m \hat{\phi}^m\,,} \tag{5.32}$$

for arbitrary power m fulfills the conditions in Eqs. (5.22), (5.24) and (5.30). Note that properly placed covariant derivatives could also be incorporated.

We can elaborate on this discussion. Suppose we know of the existence of particles with charge $+|q|$. This means that we can measure the charge of the particle (we may think of them interacting with photons). Consequently, it is an interacting theory. The only way to have an interacting relativistic theory is that there are particles with the same mass and opposite charge, $= -|q|$, which we refer to as their antiparticles.[4] The Hamiltonian/Lagrangian, or S-operator of these theories is naturally expressed in terms of $\hat{\phi} \equiv \hat{\phi}^{(+)} + \hat{\phi}_c^{(-)}$ and $\hat{\phi}^\dagger = \hat{\phi}_c^{(+)} + \hat{\phi}^{(-)}$. For the S-operator, these fields are computed in the interaction picture. These are nothing but the fields computed using the complex KG free Lagrangian studied in Sec. 4.3 using the assignments

$$\hat{a}(\mathbf{k}) \longrightarrow \hat{a}(\mathbf{k}, +|q|), \qquad \hat{b}(\mathbf{k}) \longrightarrow \hat{a}(\mathbf{k}, -|q|). \tag{5.33}$$

This interacting theory possesses a global U(1) symmetry. It is interesting to emphasize that this theory can also be formulated in terms of real scalar fields. This could be obvious, as any complex field can be written as a combination of two real fields: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ but then the interpretation is that we have two scalar particles with the same mass and the interacting theory has an $SO(2)$ symmetry.[5] The theory in terms of these fields is naturally organized in terms of vectors $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$, and the interaction terms take the form $((\phi_1, \phi_2)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix})^m$.

5.5 Cross Section

We have already provided the general expression for the S-matrix (in the following chapters we will perform perturbative computations of it). The final remaining step is to relate the S-matrix to observables. We do so in this section, which partially follows the derivation in [7].

[4] If $q = 0$ we say that particle and antiparticle are the same particle. This is described with a *real* KG field.

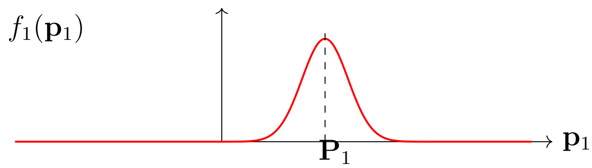
[5] Actually, for the case at hand, the symmetry is $O(2) = C \times SO(2)$, where C is the charge conjugation operator.

Initial state

As we have already mentioned, the eigenstates of momenta are physical idealizations. A more realistic two-particle state would have the following form:

$$|i\rangle = \int d\tilde{p}_1 \, d\tilde{p}_2 \, f_1(\mathbf{p}_1) \, f_2(\mathbf{p}_2) \, |\mathbf{p}_1, \mathbf{p}_2\rangle \,. \tag{5.34}$$

- For simplicity, we take two spin-zero particles with different masses.
- Wave packets f_1, f_2 are heavily weighted around $\mathbf{P}_1, \mathbf{P}_2$.



- Up to now, we have been working with creation/annihilation operators in momentum space. Let's start thinking how things look in position space by using the real KG field $\phi(x)$.
- **Case of a single particle (wave function)**

  We single out one of the particles of the $|i\rangle$ state

$$|\psi\rangle \equiv \int d\tilde{p} f(\mathbf{p}) \, |\mathbf{p}\rangle \tag{5.35}$$

  and compute

$$\begin{aligned} \psi(x) &= \langle 0| \, \hat{\phi}(x) \, |\psi\rangle = \langle 0| \, \hat{\phi}(x) \int d\tilde{p} f(\mathbf{p}) \, |\mathbf{p}\rangle \\ &= \int d\tilde{p} f(\mathbf{p}) \, \langle 0| \, \hat{\phi}(x) \hat{a}^\dagger(\mathbf{p}) \, |0\rangle = \int d\tilde{p} f(\mathbf{p}) \, e^{-ip\cdot x}. \end{aligned} \tag{5.36}$$

- We next see how this relates to $\langle\psi| \, \hat{N} \, |\psi\rangle$, i.e. to the matrix elements of the number operator.

Number operator

The (average) number of particles (of type i) is given by the expectation value of the number operator

$$N = \langle\psi| \, \hat{N}_i \, |\psi\rangle = \int d\tilde{p} \, d\tilde{p}' \, f^*(\mathbf{p}') f(\mathbf{p}) \, \langle\mathbf{p}'| \, \hat{N}_i \, |\mathbf{p}\rangle \,. \tag{5.37}$$

For each particle, we have

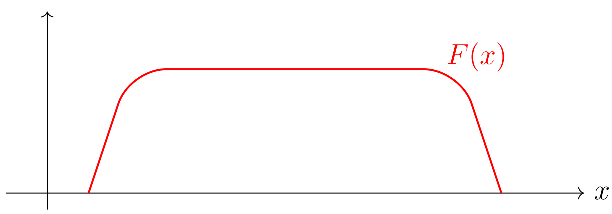
$$\langle\mathbf{p}'| \, \hat{N}_i \, |\mathbf{p}\rangle = 2\omega_{\mathbf{p}} (2\pi)^d \delta^{(d)}(\mathbf{p} - \mathbf{p}'). \tag{5.38}$$

So,

$$
\begin{aligned}
N &= \int d\tilde{p}\, |f(\mathbf{p})|^2 = i \int d^d\mathbf{x} \psi^* \overleftrightarrow{\partial}_t \psi \\
&= i \int d^d\mathbf{x} \int d\tilde{p} \int d\tilde{p}' f^*(\mathbf{p}) f(\mathbf{p}') \left(e^{-ipx} (\partial_t e^{-ip'x}) - (\partial_t e^{ipx}) e^{-ipx} \right) \\
&= i \int d^d\mathbf{x} \int d\tilde{p} \int d\tilde{p}' f^*(\mathbf{p}) f(\mathbf{p}') (-i)(\omega_{p'} + \omega_p) e^{i(p-p')x}.
\end{aligned}
$$

The wave function is conveniently written as

$$
\psi(x) = e^{-ip\cdot x} F(x), \quad \text{with } F(x) \text{ almost constant in the region of interaction.} \tag{5.39}
$$



We then get

$$
|\psi(x)|^2 = \psi^*(x)\psi(x) = |F(x)|^2, \tag{5.40}
$$

and the number density becomes

$$
\rho(x) = i\psi^*(x) \overleftrightarrow{\partial}_0 \psi = 2\omega_{\mathbf{p}} |\psi|^2 + \mathcal{O}(F'), \tag{5.41}
$$

where $|\psi|^2$ is basically constant in the region of interaction and F' is small.

We can also define

$$
n = \frac{N}{V}, \tag{5.42}
$$

as the number of particles per unit volume. Since $\rho(x)$ is approximately constant in the interaction region, we have that $n \approx \rho(x)$, and n represents the density of states per unit volume.

In the Lab frame, if we take $E = m$, we get (we reintroduce index 1 for the incoming particle)

$$
\rho_{\text{rest}} = 2m_1 |\psi_1(x)|^2, \tag{5.43}
$$

and the flux of incident particles reads

$$
\Phi = |\mathbf{v}|\rho = \left(\frac{|\mathbf{p}_2|}{E_2} \right) 2m_2 |\psi_2(x)|^2 = 2|\mathbf{p}_2||\psi_2(x)|^2. \tag{5.44}
$$

The discussion can also be extended to other spins (see [7]).

Computation of the transition probability $i \to f$

We define the probability density as

$$W_{f\leftarrow i} \equiv |\langle f, \mathrm{out}|i, \mathrm{in}\rangle|^2 \,. \tag{5.45}$$

Recall that

$$\langle f, \mathrm{out}|i, \mathrm{in}\rangle = \langle f|\hat{S}|i\rangle, \tag{5.46}$$

where $|i\rangle$ has been defined in Eq. (5.34) and $|f\rangle = |\mathbf{q}_1 \ldots \mathbf{q}_N\rangle$.

We distinguish between the non-interacting part ($\mathbb{I}$) and the purely interacting term ($i\hat{T}$)

$$\hat{S} = \mathbb{I} + i\hat{T}. \tag{5.47}$$

We take $|f\rangle \neq |i\rangle$, since in realistic experiments forward scattering is not measured. Then, the identity part $\mathbb{I}$ gives no contribution.

Due to translational invariance

$$\langle f|\hat{T}|\mathbf{p}_1, \mathbf{p}_2\rangle = (2\pi)^D \delta^{(D)}(P_f - p_1 - p_2)\langle f|\mathcal{M}|\mathbf{p}_1, \mathbf{p}_2\rangle \,. \tag{5.48}$$

Thus, we have

$$\begin{aligned} W_{f\leftarrow i} \;=\; & \int d\tilde{p}_1\, d\tilde{p}_2\, d\tilde{p}'_1\, d\tilde{p}'_2\, f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) f_1(\mathbf{p}'_1) f_2(\mathbf{p}'_2) \\ & (2\pi)^D \delta^{(D)}(P_f - p_1 - p_2)\, (2\pi)^D \delta^{(D)}(p'_1 - p'_2 - P_f) \\ & \langle f|\mathcal{M}|\mathbf{p}_1, \mathbf{p}_2\rangle^* \langle f|\mathcal{M}|\mathbf{p}'_1, \mathbf{p}'_2\rangle \,, \end{aligned} \tag{5.49}$$

where

$$(2\pi)^D \delta^{(D)}(p_1 + p_2 - p'_1 - p'_2) = \int d^D x \, e^{i(p_1+p_2-p'_1-p'_2)\cdot x}. \tag{5.50}$$

Then,

$$\begin{aligned} W_{f\leftarrow i} \;=\; & \int d^D x\, d\tilde{p}_1\, d\tilde{p}_2\, d\tilde{p}'_1\, d\tilde{p}'_2\, f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) f_1(\mathbf{p}'_1) f_2(\mathbf{p}'_2) \\ & \times \langle f|\mathcal{M}|\mathbf{p}_1, \mathbf{p}_2\rangle^* \langle f|\mathcal{M}|\mathbf{p}'_1, \mathbf{p}'_2\rangle e^{-ip'_1\cdot x} e^{-ip'_2\cdot x} e^{ip_1\cdot x} e^{ip_2\cdot x} \\ & \times (2\pi)^D \delta^{(D)}(P_f - p_1 - p_2) \,. \end{aligned} \tag{5.51}$$

Note that $f_1(\mathbf{p}_1)$ is collimated around $\mathbf{P}_1$, so we can work as if

$$f_1(\mathbf{p}_1) \sim \delta(\mathbf{p}_1 - \mathbf{P}_1) \,, \tag{5.52}$$

and use Eq. (5.36). Thus,

$$\boxed{W_{f\leftarrow i} \simeq \int d^D x \, |\psi_1(x)|^2 |\psi_2(x)|^2 \, (2\pi)^D \delta^{(D)} \left(P_f - P_1 - P_2\right) |\langle f|\mathcal{M}|\mathbf{P}_1, \mathbf{P}_2\rangle|^2 \,.} \tag{5.53}$$

Luminosity Density

We define the luminosity as

$$L = \int d^D x \, |\psi_1(x)|^2 |\psi_2(x)|^2,$$

and the luminosity density as

$$\mathcal{L} = |\psi_1(x)|^2 |\psi_2(x)|^2.$$

This quantity is specific to the experimental setup. Instead, we would prefer to work with quantities that are independent of the particular characteristics of the experiment being used. We discuss this in what follows.

Probability

The probability that the initial state i goes to a bunch of final states f for a portion Δ_f of the phase space of each f channel reads

$$P_\Delta = \sum_f \int_{\Delta_f} dQ \, W_{f\leftarrow i}, \tag{5.54}$$

where $|f\rangle = |\mathbf{q_1} \dots \mathbf{q_n}\rangle$, $\sum\limits_f$ is the sum over a partial set of channels, and $dQ = \prod\limits_{i=1}^{n} d\tilde{q}_i$ refers to the integration measure over the three-momenta of the final state particles.

We can then define the differential probability for a given channel as

$$dP = dQ W_{f\leftarrow i} \, . \tag{5.55}$$

The fact that $|\psi_1(x)|^2 |\psi_2(x)|^2$ is approximately x-independent in the interaction region, naturally leads us to define the density probability per space-time volume unit:

$$\frac{dP}{dV\,dt} \equiv [\rho_{\text{target}}] \, [\text{flux factor}] \, d\sigma \, , \tag{5.56}$$

where $(\langle f|\mathcal{M}|i\rangle = \langle q_1 \dots q_N|\mathcal{M}|\mathbf{P_1}, \mathbf{P_2}\rangle)$

$$\boxed{d\sigma = (2\pi)^D \delta^{(D)}(P_f - P_1 - P_2) \frac{|\langle f|\mathcal{M}|i\rangle|^2}{4m_1|\mathbf{P_2}|} dQ.} \tag{5.57}$$

This separates the experimental setup from machine-independent quantities.

The above derivation has been done in the Lab frame. This information is encoded in the factor $4m_1|\mathbf{P_2}|$. We can write this factor in an explicit Lorentz invariant way by using the following equality ($s = (P_1 + P_2)^2$):

$$4m_1|\mathbf{P}_2| = 4m_1\sqrt{E_2^2 - m_2^2} = 4\sqrt{(P_1 \cdot P_2)^2 - m_1^2 m_2^2} = 2\sqrt{\lambda(s, m_1^2, m_2^2)}\,. \quad (5.58)$$

The last equality is explicitly Lorentz invariant, and can be written in terms of the *Källén function*:

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \quad (5.59)$$

In the center-of-mass frame, it simplifies to

$$4\sqrt{E^4 - m^4} \simeq s \to \sqrt{s} \gg \sqrt{m^2} \quad \text{(high energies)}. \quad (5.60)$$

The differential cross section is defined in a general reference frame as

$$\boxed{d\sigma = \frac{1}{2\,\lambda^{1/2}(s, m_1^2, m_2^2)}(2\pi)^D\,\delta^{(D)}(P_f - P_1 - P_2)\,|\langle f|\mathcal{M}|i\rangle|^2 dQ\,.} \quad (5.61)$$

The total cross section reads

$$\boxed{\sigma = \frac{(2\pi)^D}{2\,\lambda^{1/2}(s, m_1^2, m_2^2)}\frac{1}{n!}\overline{\sum_{pol(1,2)}}\sum_{\text{pol}(f)}\int dQ\,\delta^{(D)}(P_f - P_1 - P_2)\,|\langle f|\mathcal{M}|i\rangle|^2\,.} \quad (5.62)$$

Here, the notation $\overline{\sum\limits_{pol(1,2)}}$ represents the average sum (via the density matrix) over the polarizations of the initial 1 and 2 particles. This averaging is necessary when the preparation of the initial state is incomplete—that is, when only the momenta are specified, but not the spin of the states. $dQ = d\tilde{q}_1 d\tilde{q}_2 \ldots$ represents the Lorentz-invariant phase space integration measure of the final-state momenta. The factor $\frac{1}{n!}$ is included in order to avoid double counting should we have n identical particles in the final state. If there are multiple sets of identical particles, a separate factor should be introduced for each set, such as $\frac{1}{n_1!}$, $\frac{1}{n_2!}$, and so on.

The expressions above depend on the normalization used for the incoming particles. The present derivation has used Eq. (3.29), the normalization for bosons. For fermions the normalization is different. The resulting expressions in this case can be found in Appendix A.3 of [7].

5.5.1 Decay Width (rest frame)

The derivation is equal to the cross section but the initial state only has one particle.

- $|i\rangle = \int d\tilde{p}_a\, f(\mathbf{p}_a)|\mathbf{p}_a\rangle$

$$P_\Delta = \sum_f \int_\Delta dQ\, W_{f\leftarrow i}, \qquad dP = dQ\, W_{f\leftarrow i} \tag{5.63}$$

$$\begin{aligned}\frac{dP}{dx} &= |\psi(x)|^2 (2\pi)^D \delta^{(D)}(P_f - P_a)\, |\langle f|\mathcal{M}|\mathbf{P}_a\rangle|^2\, dQ \\ &= 2m_a |\psi(x)|^2 (2\pi)^D \delta^{(D)}(P_f - P_a)\, |\langle f|\mathcal{M}|\mathbf{P}_a\rangle|^2\, dQ \frac{1}{2m_a} \\ &= \rho\, d\Gamma,\end{aligned} \tag{5.64}$$

where $\rho = 2m_a|\psi(x)|^2$ and

$$d\Gamma = (2\pi)^D \delta^{(D)}(P_f - P_a)\, |\langle f|\mathcal{M}|\mathbf{P}_a\rangle|^2\, dQ \frac{1}{2m_a}\,. \tag{5.65}$$

The decay width for a particular channel reads

$$\Gamma(a \to n) = \frac{(2\pi)^D}{2m_a} \overline{\sum_{\text{pol a}}} \sum_{\text{pol f}} \int \prod_{i=1}^n d\tilde{q}_i\ \ \delta^{(D)}\left(P_a - \sum_{i=1}^n q_i\right)\ |\langle f|\mathcal{M}|i\rangle|^2\, \frac{1}{m!}, \tag{5.66}$$

where $\overline{\sum}_{\text{pol a}}$ is the average over polarizations of the initial particles. It has to be introduced if we do not know its initial polarization. $\sum_{\text{pol f}}$ corresponds to the sum of the polarizations of the final states. It has to be included if we do not measure the polarizations of the final states, or if we measure them and sum over all possible states with different polarizations. As in Eq. (5.62), we have to add $\frac{1}{m!}$ factors to the total decay width for each set of m identical particles we have in the final state (note that the total number of final particles could be different to the number of final identical particles, or that we could have different subsets of identical particles).

The total decay width is the sum of the decay widths of all possible channels.

$$\Gamma(a \to \text{all}) = \sum_{\text{channels}} \Gamma(a \to n). \tag{5.67}$$

Finally, branching ratios are defined as

$$\mathcal{B}(a \to n) = \frac{\Gamma(a \to n)}{\Gamma(a \to \text{all})}. \tag{5.68}$$

5.5.2 Phase Space Integrals

When considering cross sections or decays, we will have to compute integrals over the phase space of the particles of the final state. In this book, we will typically consider two-particle final states. We will always assume that we have azimuthal symmetry. For the cross-section, we will have the following configuration (p_1 and p_2 are incoming, and p_3 and p_4 outgoing):

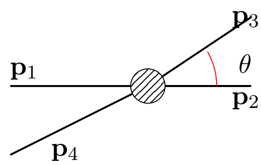


Figure 5.1: Symbolic representation of a 1+2 → 3+4 scattering.

where the lines could represent equal or different particles. Generally, the matrix element square will depend on θ only. If the matrix element square is independent of the angle of emission, the phase space integral factors out, and merely corresponds to performing the following integral:

Exercise

Compute the following integral ($p_i^2 \equiv m_i^2 > 0$):

$$I_1 = \int \frac{d^d\mathbf{p}_2}{(2\pi)^d 2E_2} \frac{d^d\mathbf{p}_3}{(2\pi)^d 2E_3} \delta^{(D)}(p_1 - p_2 - p_3) \,.$$

Hint: The integral is invariant under orthochronous Lorentz transformations.

This integral is considered in Sec. 6.1.1, where we consider the scattering of two mesons at tree level in what we name a $\lambda\phi^4$ theory.

I_1 is also the integral that appears in Eq. (5.66) for the decay into two particles if the initial state spin does not have a privileged direction, either because it is a scalar or because we average over the polarizations of the initial particle.

If the matrix element square depends on θ, we cannot use I_1, but we can still partially perform the integration over the phase space. One then obtains the differential cross section over the differential solid angle, which one traditionally writes as

$$\frac{d\sigma}{d\Omega} \,. \tag{5.69}$$

Examples of these computations in the center-of-mass frame and in the Lab frame can be found in Secs. 6.2.1 and 10.2.1 respectively.

To avoid the frame dependence of these results, one may also consider writing the differential cross section in terms of $t = (p_1 - p_3)^2$, rather than in terms of θ. The result would then be explicitly Lorentz invariant. The differential cross sections in terms of t can be written in the following way:

$$\boxed{\begin{aligned}\frac{d\sigma}{dt} = \frac{(2\pi)^D}{2\,\lambda^{1/2}(s, m_1^2, m_2^2)} \overline{\sum_{pol(1,2)}} \sum_{\mathrm{pol}(f)} \int d\tilde{p}_3\, d\tilde{p}_4 \delta^{(D)}(p_3 + p_4 - p_1 - p_2) \\ \times \delta(t - (p_1 - p_3)^2)\, |\langle f|\mathcal{M}|i\rangle|^2\,,\end{aligned}} \tag{5.70}$$

and the total cross section would read (up to a symmetrization factor for equal final particles)

$$\sigma = \int dt \frac{d\sigma}{dt}\,. \tag{5.71}$$

The integral over the phase space can actually be performed by using the fact that

Exercise

$$\begin{aligned} I_2 &= \int \frac{d^d\mathbf{p}_3}{(2\pi)^d 2E_3} \frac{d^d\mathbf{p}_4}{(2\pi)^d 2E_4} \delta^4(p_1 + p_2 - p_3 - p_4)\delta(t - (p_1 - p_3)^2) \\ &= \frac{\pi}{2(2\pi)^d} \frac{1}{\lambda^{1/2}(s, m_1^2, m_2^2)}\,. \end{aligned} \tag{5.72}$$

This allows us to write Eq. (5.70) in a more compact way:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} \frac{1}{\lambda(s, m_1^2, m_2^2)} \overline{\sum_{pol(1,2)}} \sum_{\mathrm{pol}(f)} |\langle f|\mathcal{M}|i\rangle|^2\,. \tag{5.73}$$

6 Interaction (examples): $\lambda\phi^4$ and $\lambda\phi^3$

[1] We consider an interacting theory between scalar particles (one may think of these scalars as Higgs bosons or some hadronic mesons). We have

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$

with

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2, \qquad \mathcal{L}_I = F(\phi). \tag{6.1}$$

We restrict ourselves to the case where $\mathcal{L}_I$ is a purely self-interacting term and contains no time derivatives. Therefore, if the theory is Poincaré invariant, the interaction must be a pure function of ϕ (typically a polynomial). In this situation, we can easily go to the Hamiltonian formalism. The reason is that the canonical conjugate π of the interacting theory is the same as the free field case:

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}_0}{\partial\dot{\phi}} = \pi_0 \,. \tag{6.2}$$

This basically makes the relation between the interaction Hamiltonian term and the interaction Lagrangian term trivial:

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}_0 - \mathcal{L}_I = H_0 - \mathcal{L}_I \Rightarrow \boxed{\mathcal{H}_I = -\mathcal{L}_I}\,, \tag{6.3}$$

so the S-matrix operator reads

$$\boxed{S = T\left\{e^{i\int d^4x\mathcal{L}_I^I(x)}\right\}}, \quad \text{where} \quad \mathcal{L}_I^I(x) = F(\phi_I(x)). \tag{6.4}$$

As we mentioned in the previous chapter, $\phi_I(x) = e^{iH_0x^0}\phi(\mathbf{x})e^{-iH_0x^0}$ are nothing but the free fields we obtained in previous chapters for the real KG field:

$$\phi_I(x) = \int d\tilde{k}\left(e^{-ik\cdot x}a(\mathbf{k}) + e^{ik\cdot x}a^\dagger(\mathbf{k})\right). \tag{6.5}$$

The Lagrangian density $\mathcal{L}_I(\phi_I)$ has all possible combinations of $\hat{a}$ and $\hat{a}^\dagger$. We want to calculate $\langle\beta|S|\alpha\rangle$ where $|\alpha\rangle = a^\dagger a^\dagger \ldots |0\rangle$ and $\langle\beta| = \langle 0|\, aa...$. Therefore, a generic contribution to the S-matrix would have the following structure:

$$\langle 0|(aa\cdots a)\;[aa^\dagger\cdots aaa^\dagger]\;(a^\dagger a^\dagger\cdots a^\dagger)|0\rangle\,. \tag{6.6}$$

[1] For simplicity, we omit introducing the $\widehat{\ }$ notation for the operators in this chapter, and also explicitly set $D = 4$ in the rest of the book.

The point is to look for non-zero contributions to the S-matrix and to have an efficient way to compute them. In this respect the following result is important:

Exercise

Prove that

$$\langle 0|\underbrace{aa\cdots a}_{m}\underbrace{a^\dagger\cdots a^\dagger}_{n} = 0 \qquad m < n\,; \tag{6.7}$$

$$\underbrace{aa\cdots a}_{m}\underbrace{a^\dagger\cdots a^\dagger}_{n}|0\rangle = 0 \qquad m > n\,; \tag{6.8}$$

$$\langle 0|\underbrace{aa\cdots a}_{m}\underbrace{a^\dagger\cdots a^\dagger}_{n}|0\rangle = 0 \quad \text{if } m \neq n\,. \tag{6.9}$$

Therefore, it turns out to be convenient to write the interaction terms in a normal ordering way: $\sim (a^\dagger\cdots a^\dagger)(a\cdots a)$. This is what Wick's theorem does (see Sec. 4.4). It relates T-ordered operators with normal-ordered operators (plus contractions).

6.1 Meson-meson Scattering ($\lambda\phi^4$)

We now consider a particular interaction term:

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4, \qquad \mathcal{H}_I = \frac{\lambda}{4!}\phi^4, \qquad \mathcal{H}_I^I = \frac{\lambda}{4!}\phi_I^4, \tag{6.10}$$

ϕ_I is given by Eq. (6.5), the KG free field (we will omit the I index in the following).

We aim to compute the elastic meson-meson scattering at tree level. The initial and final states read

$$|i\rangle = a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle, \qquad |f\rangle = a^\dagger(\mathbf{p}_3)a^\dagger(\mathbf{p}_4)|0\rangle, \tag{6.11}$$

with $\mathbf{p}_3, \mathbf{p}_4 \neq \mathbf{p}_1, \mathbf{p}_2$ (we assume there is interaction). The S-matrix reads

$$\begin{aligned} S_{f\leftarrow i} &= \langle f|T\{e^{-i\frac{\lambda}{4!}\int d^4x\phi^4(x)}\}|i\rangle = \langle f|i\rangle - i\frac{\lambda}{4!}\langle f|\int d^4x\phi^4(x)|i\rangle + O(\lambda^2) \\ &\simeq -i\frac{\lambda}{4!}\langle f|\int d^4x(\phi^{(+)}+\phi^{(-)})(\phi^{(+)}+\phi^{(-)})(\phi^{(+)}+\phi^{(-)})(\phi^{(+)}+\phi^{(-)})|i\rangle. \end{aligned} \tag{6.12}$$

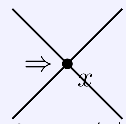
We will use this computation as a first template to illustrate how a diagrammatic representation of the computation can emerge.

Feynman Rules in Position Space (ϕ^4 Theory)

1. Fundamental Diagram Components

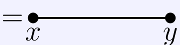
- **Interaction Vertex:** The interaction term $\phi^4(x)$ in the Lagrangian corresponds to a vertex where four lines meet. Symbolically, we draw it as below. Nevertheless, the Feynman rule corresponds only to the dot. We put the lines for convenience, to indicate how many lines it has to be contracted with.

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4(x) \quad \Rightarrow$$

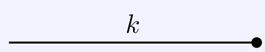


- **Propagator:** The contraction of two fields, $\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$, corresponds to an internal line (the Feynman propagator $i\Delta_F(x-y)$) connecting two vertices, x and y.

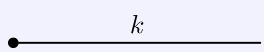
$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle =$$



- **External particles:** An external incoming particle will be represented in the following way



and, similarly, for an external outgoing particle:



2. Scattering Diagram. A S-matrix $\langle f|S|i\rangle$ is given by the sum of Feynman diagrams. Each one of them is represented by connecting vertices, propagators and external particles. The latter are fixed for a given physical process (initial state $|i\rangle$ and final state $\langle f|$).

Important Note ("No dots alone!"): The dot of the vertex has four "legs", one for each ϕ field of the $\phi^4(x)$ vertex. Each field looks for something to contract with. The contraction can be either with the asymptotic states or with other fields of the interaction terms. In both cases legs are generated. In the latter case, these legs become propagators, in the former case external legs.

A diagram is **connected** if all its parts are joined. The relevant contributions to the S-matrix come from connected diagrams (recall that the initial and final states are different).

We now proceed to compute Eq. (6.12). Since we are working to $O(\lambda)$, all fields are located at the same point in space-time and we cannot (naively) apply Wick's theorem. Still, it is possible to organize the computation in terms of normal-ordered contributions plus/times propagators. The qualification in this case is that the propagators are computed at zero distance. One might wonder how Feynman propagators may show up, since there is no time-ordered computation. The reason behind lies in the following exercise:

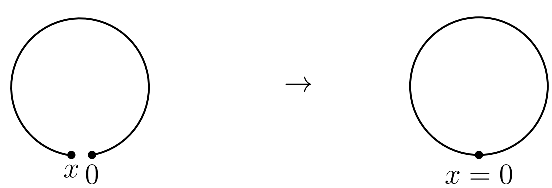
Exercise

Show that

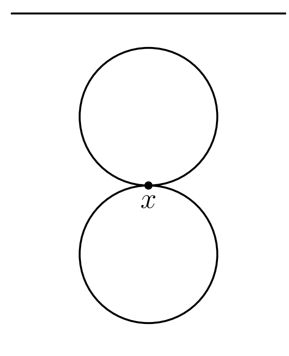
$$\lim_{x\to 0}\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(0) = \lim_{x\to 0}\langle 0|T\{\phi(x)\phi(0)\}|0\rangle \tag{6.13}$$
$$= [\phi^{(+)}(0),\phi^{(-)}(0)] = \int_{\mathbb{R}^4} d^4k\frac{i}{k^2-m^2+i\eta} = \infty.$$

As follows from this exercise, this quantity is infinite. This is a problem. Nevertheless, let us see if this is a problem we have to address in this book.

The commutators appear in the computation when trying to get a normal-ordered expression. Then, using the result of the previous exercise, one can relate these commutators at zero distance with Feynman propagators at zero distance. The diagrammatic interpretation of $\lim_{x\to 0}\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(0)$ is a propagator where the initial and final points are located at the same point. This is represented by a closed loop. Pictorially, what we have is the following:

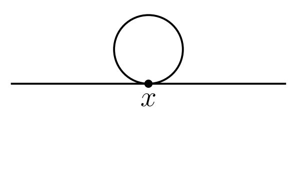


These objects appear wherever we have contractions within the vertex. We have two possibilities at $\mathcal{O}(\lambda)$. One is to have two contractions from terms like $\contraction{}{\phi}{\phi}{\phi}\contraction[2ex]{\phi}{\phi}{\phi}{\phi}\phi\phi\phi\phi$. This generates a constant term in the interaction Hamiltonian: $\delta H_I \sim$ *c-number* (infinite). The contribution to the matrix element would be *number* $\times$ $\langle f|i\rangle$. This is zero when the final state is different from the initial one. The *number* is, however, infinite. This is a problem. The actual solution to this problem goes beyond the scope of this book. Nevertheless, we briefly outline the idea. The procedure is to first regularize the theory, making the *number* finite but dependent on a parameter that diverges as it approaches its physical value. On the other hand, $\langle f|i\rangle$ remains exactly zero. Diagrammatically, the contribution to the S-matrix would be something like



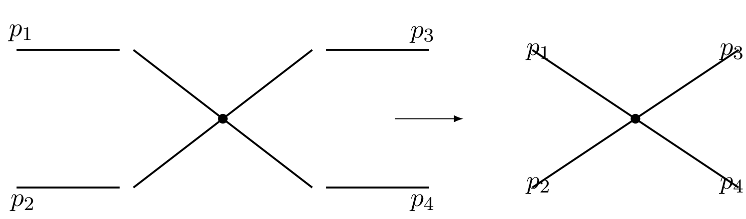
Leaving aside the external legs, this is an example of a vacuum diagram.

Terms with a single contraction will not contribute to the S-matrix either. They generate a correction to the interaction Hamiltonian that has the following form: $\delta H_I \sim \textit{number} \times (a^\dagger a^\dagger + aa + aa^\dagger + a^\dagger a)$. The first two terms directly give directly zero because of Eq. (6.9). The other two generate corrections that, pictorially, can be drawn in the following form:



which is zero if the final state is different from the initial state.

We now consider the contraction of the four fields of the interaction vertex with the external particles. The following connected diagram is the only possibility. The entire interaction occurs at a single point.



We have to consider all possible (non-vanishing) contractions. We need to have an equal number of $\phi^{(+)}$ and $\phi^{(-)}$ (in our case, two of each). Therefore,

$$S_{f\leftarrow i} = -i\frac{\lambda}{4!}\binom{4}{2}\langle f| \int d^4x : \phi^{(-)}(x)\phi^{(-)}(x)\phi^{(+)}(x)\phi^{(+)}(x) : |i\rangle\,, \tag{6.14}$$

where we have already placed them in the convenient order. This generates a factor $\binom{4}{2} = 4!/(2!2!) = 6$ that comes from combinatorics. In terms of creation and annihilation operators, we have

$$\begin{aligned} S_{f\leftarrow i} &= -i\frac{\lambda}{4}\langle 0|a(\mathbf{p}_3)a(\mathbf{p}_4) \\ &\left(\int d^4x \int d\tilde{k}_1 a^\dagger(\mathbf{k}_1)e^{ik_1x}\int d\tilde{k}_2\cdots\int d\tilde{k}_3 a(\mathbf{k}_3)e^{-ik_3x}\int d\tilde{k}_4\ldots\right) \\ &a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle\,. \end{aligned} \tag{6.15}$$

We then need to compute objects such as the following:

$$\langle 0|a(\mathbf{p}_3)a(\mathbf{p}_4)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p_2})|0\rangle\,. \tag{6.16}$$

This is done by moving the annihilation operators of the interaction term to the right and the creation to the left, until both reach the vacuum where they vanish. In the process, we use the commutation relations. For example:

$$a(\mathbf{k}_4)a^\dagger(\mathbf{p}_1) = a^\dagger(\mathbf{p}_1)a(\mathbf{k}_4) + (2\pi)^3 2\omega_{p_1}\delta^{(3)}(\mathbf{k}_4 - \mathbf{p}_1). \tag{6.17}$$

After commuting all annihilation operators to the right until they hit $|0\rangle$, the only non-vanishing terms are those where every a is contracted with an $a^\dagger$. There are 2 ways to contract the two $\phi^{(+)}$ with the initial state particles and two ways to contract $\phi^{(-)}$ with the final state particles. For the initial state particles, the two combinations would read

$$\begin{aligned} a(\mathbf{k}_3)a(\mathbf{k}_4)a^\dagger(\mathbf{p}_1)\hat{a}^\dagger(\mathbf{p}_2)|0\rangle &\to (2\pi)^3 2\omega_{p_1}\delta^{(3)}(\mathbf{k}_4 - \mathbf{p}_1)(2\pi)^3 2\omega_{p_2}\delta^{(3)}(\mathbf{k}_3 - \mathbf{p}_2)|0\rangle\,, \\ a(\mathbf{k}_3)a(\mathbf{k}_4)a^\dagger(\mathbf{p}_1)\hat{a}^\dagger(\mathbf{p}_2)|0\rangle &\to (2\pi)^3 2\omega_{p_1}\delta^{(3)}(\mathbf{k}_3 - \mathbf{p}_1)(2\pi)^3 2\omega_{p_2}\delta^{(3)}(\mathbf{k}_4 - \mathbf{p}_2)|0\rangle\,. \end{aligned} \tag{6.18}$$

After integration over $\mathbf{k}_3$ and $\mathbf{k}_4$, both terms give the same contribution. For example,

$$\int \frac{d^d\mathbf{k}_3}{(2\pi)^3 2\omega_{k_3}} e^{-ik_3x}(2\pi)^3 2\omega_{k_3}\delta^{(3)}(\mathbf{p}_2 - \mathbf{k}_3) = e^{-ip_2x}. \tag{6.19}$$

This is general. For incoming particles, we get e^{-ip_ix}, and for outgoing particles, we get e^{+ip_fx}. The final result is thus

$$\begin{aligned} S_{f\leftarrow i} &= -i\frac{\lambda}{4}\cdot 2\cdot 2\int d^4x(e^{ip_3x}e^{ip_4x}e^{-ip_1x}e^{-ip_2x}) \\ &= (-i\lambda)(2\pi)^4\delta^{(4)}(p_3 + p_4 - p_1 - p_2)\,, \end{aligned} \tag{6.20}$$

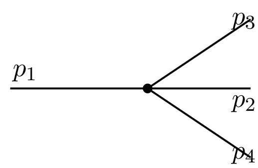
and,

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(\Sigma p)\langle f|\mathcal{M}|i\rangle. \tag{6.21}$$

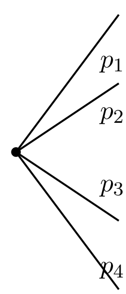
Then,

$$\langle f|\mathcal{M}|i\rangle = -\lambda. \tag{6.22}$$

The above discussion applies to the elastic meson-meson scattering. However, one of the beauties of QFTs applied to relativistic systems is that the same interaction vertex relates different processes (different particle content in the initial and final states). At $\mathcal{O}(\lambda)$, and within the context of this theory, the range of possibilities is quite limited. Still, we would like to emphasize that the same vertex can generate diagrams such as



or



The first example is one scalar decaying into (the same type of) three scalars. The second is the generation of four scalars out of the vacuum. Both examples yield zero due to energy-momentum conservation.[2] Note though, that this would not necessarily be so if we are in a medium at finite density or temperature. There, the medium can supply the necessary energy to produce these particles out of the vacuum. Nevertheless, we will not study those situations in this book.

6.1.1 Cross-section for Meson-meson Scattering

Now, let us compute the (differential) cross-section:

$$\begin{aligned}\int_\Delta d\sigma &= \frac{1}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}} \\ &\times \int_\Delta d\tilde{p}_3 d\tilde{p}_4 |\langle p_3, p_4|\mathcal{M}|p_1, p_2\rangle|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4).\end{aligned} \tag{6.23}$$

[2]The first example could yield a non-zero result if the interaction term were of the form $\phi_1\phi_2^3$ and $m_1 > 3m_2$.

In our case $m_1 = m_2 = m$. Δ stands for a fraction of the phase space such that there is no overlap in the phase space of $\mathbf{p}_3$ and $\mathbf{p}_4$.

In the distant past, we have two free particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, and in the distant future, two free particles with momenta $\mathbf{p}_3$ and $\mathbf{p}_4$. We represented this situation in Fig. 5.1. Here, there is a potential problem of interpretation: It is not possible to distinguish between particles 3 and 4. We only know that one particle comes out with an angle θ, and the other comes out in the opposite direction with opposite momentum in the center-of-mass frame. Therefore, what we really measure is the integral

$$\int d\tilde{p}_3 \int d\tilde{p}_4 \to \int_{\Delta_3} d\tilde{p}_3 \int_{\Delta_4} d\tilde{p}_4 + \int_{\Delta_4} d\tilde{p}_3 \int_{\Delta_3} d\tilde{p}_4. \tag{6.24}$$

As a result, we have to put a symmetry factor 1/2. Alternatively, we can perform the computation without introducing any symmetry factor, but then we must be careful with the phase space integral to avoid double counting. In other words, we cannot integrate over the whole phase space. This boils down to the fact that $|p_3, p_4\rangle$ and $|p_4, p_3\rangle$ are the same state and we sum over both of them when we integrate over the whole space, producing a double counting.

Let us now compute the phase space integral. The matrix element has spherical symmetry. This will help us to calculate the integral. We calculate it in the center-of-mass frame (we can do it in any frame because the integral is Lorentz invariant).

$$\int d\tilde{p}_4 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) = (2\pi)\delta((p_1 + p_2 - p_3)^2 - m^2)\theta(p_1^0 + p_2^0 - p_3^0). \tag{6.25}$$

In the center-of-mass frame, $p_1 = (E, \mathbf{p})$, $p_2 = (E, -\mathbf{p})$. The prefactor in Eq. (6.23) then simplifies to

$$\begin{aligned}(p_1 p_2)^2 - m^4 &= (E^2 + \mathbf{p}^2)^2 - m^4 = (2E^2 - m^2)^2 - m^4 = 4E^4 - 4E^2 m^2 \\ &= 4E^2(E^2 - m^2) = 4E^2 |\mathbf{p}|^2.\end{aligned} \tag{6.26}$$

Therefore,

$$d\sigma = \frac{1}{8E|\mathbf{p}|} \int \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2\omega_3} (2\pi)\lambda^2 \delta((p_1 + p_2 - p_3)^2 - m^2)\theta(p_1^0 + p_2^0 - p_3^0). \tag{6.27}$$

But $p_1^0 + p_2^0 - p_3^0 = 2E - \omega_3 > 0$ since $p_1^0 + p_2^0 = p_3^0 + p_4^0 > p_3^0$. By writing the $\mathbf{p}_3$ integral in spherical coordinates, we have

$$\frac{d^3 p_3}{\omega_3} = \frac{|\mathbf{p}_3|^2 d|\mathbf{p}_3|}{\omega_3} d\Omega_3 = |\mathbf{p}_3| d\omega_3 d\Omega_3, \quad \text{since } d\omega_3 = d(\sqrt{m^2 + |\mathbf{p}_3|^2}) = \frac{|\mathbf{p}_3|}{\omega_3} d|\mathbf{p}_3|. \tag{6.28}$$

Additionally,

$$\delta((p_1+p_2-p_3)^2-m^2) = \delta((2E-\omega_3)^2-\mathbf{p}_3^2-m^2) = \delta(4E^2-4E\omega_3) = \frac{1}{4E}\delta(E-\omega_3). \tag{6.29}$$

Then,

$$d\sigma = \frac{\lambda^2}{(2\pi)^2 8E|\mathbf{p}|4E}\int \frac{|\mathbf{p}_3|d\omega_3 d\Omega_3}{2}\delta(E-\omega_3) = \frac{\lambda^2}{64(2\pi)^2E^2}d\Omega_3. \tag{6.30}$$

So,

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{64\pi^2 s}, \tag{6.31}$$

where $s = (p_1+p_2)^2$ is the Mandelstam variable, and Ω is the angle of any of the final particles. The integration over the solid angle can be done in two ways. We can integrate over the whole solid angle and introduce the symmetry factor 1/2. Then, the total cross section reads

$$\sigma = \frac{\lambda^2}{64\pi^2 s}\frac{4\pi}{2} = \frac{\lambda^2}{32\pi s}. \tag{6.32}$$

The other option is to integrate only over the range $2\pi\int_0^1 d(\cos(\theta))$. Obviously, one obtains the same result. Yet, this highlights that obtaining the correct phase space region to integrate over – so as to avoid double counting – becomes increasingly more complicated when dealing with more than two final-state particles.

6.2 Meson-meson Scattering ($\lambda\phi^3$)

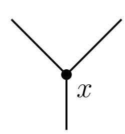
This discussion runs parallel to the one of the previous section with the qualification that now

$$\mathcal{L}_I = -\frac{\lambda}{3!}\phi^3. \tag{6.33}$$

Again, we want to calculate $S_{f\leftarrow i} = \langle f|T\{e^{i\int d^4x\mathcal{L}_I^I(x)}\}|i\rangle$, where

$$|i\rangle = \hat{a}^\dagger(\mathbf{p}_1)\hat{a}^\dagger(\mathbf{p}_2)|0\rangle \qquad \text{and} \qquad |f\rangle = \hat{a}^\dagger(\mathbf{p}_3)\hat{a}^\dagger(\mathbf{p}_4)|0\rangle.$$

The basic vertex is a 3-point vertex:



Unlike in the previous section, to $\mathcal{O}(\lambda)$, we obtain zero.

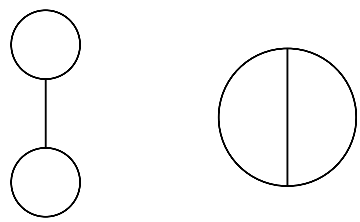
$$\langle f| : \int d^4x\mathcal{L}_I^I(x) : |i\rangle = 0 \tag{6.34}$$

This is because we do not have the same number of a and $a^\dagger$ inside the bracket once it is written in the form $\langle 0| \ldots |0\rangle$. We need to go to $\mathcal{O}(\lambda^2)$.

$$S_{f\leftarrow i} \sim \langle f|T \left\{ \frac{(-i)^2}{2!} \left(\frac{\lambda}{3!}\right)^2 \int d^4x\phi^3(x) \int d^4y\phi^3(y) \right\} |i\rangle. \tag{6.35}$$

We have to compute $T\{\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\}$, and its sandwich with the initial and final states. We organize the computation by the number of internal contractions between the vertices:

- The term with three contractions corresponds to vacuum diagrams, which do not contribute, as we saw for $\lambda\phi^4$. Examples of vacuum diagrams are

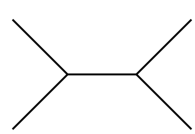


  The left-hand diagram corresponds to the case where contractions take place in the same interaction vertex. The right-hand diagram would correspond to a contraction with the following structure:
  $\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)$.

- The term with two contractions, like $\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)$, also gives zero because it leads to terms like *number* $\times \langle f|(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger + \ldots)|i\rangle = 0$.

- The term with zero contractions does not contribute either, as the number of creation and annihilation operators does not match once the expression is written in a normal-ordered form.

- The term with one contraction between two fields of the two different vertices, e.g., $\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)$:



  is the only one that will yield a non-zero contribution. We now discuss this contribution in depth.

The S-matrix element at tree level (first non-zero order) is given by connecting the two vertices with one propagator and the remaining four fields to the external legs:

$$\begin{aligned} S &\approx -\frac{1}{2!}\left(\frac{\lambda}{3!}\right)^2 \cdot 9 \int d^4x d^4y \langle p_2', p_1'| : \phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y) : |p_1, p_2\rangle \\ &= -\frac{1}{2!}\left(\frac{\lambda}{3!}\right)^2 \cdot 9 \int d^4x d^4y \times \\ &\Big\{ \langle p_2', p_1'| : \phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y) : |p_1, p_2\rangle \cdot (2\cdot 4) \\ &+ \langle p_2', p_1'| : \phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y) : |p_1, p_2\rangle \cdot (2\cdot 4) \\ &+ \langle p_2', p_1'| : \phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y) : |p_1, p_2\rangle \cdot (2\cdot 4) \Big\}, \end{aligned} \tag{6.36}$$

where the factor 9 comes from the 3×3 possible ways of contracting $\phi(x)$ with $\phi(y)$. The factor 2 comes from $x \leftrightarrow y$ symmetry and the 4 from the possible ways to contract the fields with the external state (2(incoming) $\times$ 2(outgoing) $= 4$). Recall now that $\phi(x)|\mathbf{p}_1\rangle = e^{-ixp_1}|0\rangle$. So we get:

$$\begin{aligned} S \approx (-i\lambda)^2 \int d^4x d^4y \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\eta} \\ \times \left(e^{-iy(p_1+p_2)} e^{ix(p_1'+p_2')} + e^{-iy(p_1-p_1')} e^{ix(p_2'-p_2)} + e^{-iy(p_1-p_2')} e^{ix(p_1'-p_2)}\right). \end{aligned}$$

We set $x = y + x'$ so that

$$\begin{aligned} -i\lambda^2 \int d^4x\, d^4y \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik\cdot x'}}{k^2 - m^2 + i\eta} \left(e^{-iy\cdot(p_a+p_2-p_1'-p_2')} e^{ix'\cdot(p_1+p_2)}\right. \\ \left. + e^{-iy\cdot(p_1-p_1'+p_2'-p_2)} e^{ix'\cdot(p_1-p_2)} + e^{-iy\cdot(p_1-p_1'+p_2-p_2')} e^{ix'\cdot(p_1-p_2')}\right), \end{aligned}$$

and use

$$\int d^4x'\, e^{-ik\cdot x'} e^{ix'\cdot(p_1+p_2)} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k) \doteq (2\pi)^4 \delta^{(4)}(p_1' + p_2' - k),$$

$$\int d^4x'\, e^{-ik\cdot x'} e^{ix'\cdot(p_2'-p_2)} = (2\pi)^4 \delta^{(4)}(p_2' - p_2 - k) \doteq (2\pi)^4 \delta^{(4)}(p_1 - p_1' - k).$$

Overall, after performing the integrals over x and y, which yield delta functions, and then integrating over k, we obtain:

$$\begin{aligned} S &\approx i(2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2) \\ &\times (-i\lambda)^2 \left(\frac{1}{(p_1+p_2)^2 - m^2} + \frac{1}{(p_1'-p_2)^2 - m^2} + \frac{1}{(p_1'-p_1)^2 - m^2}\right). \end{aligned} \tag{6.37}$$

In conclusion, by using Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p'_1)^2$, $u = (p_1 - p'_2)^2$, we have:

$$\langle f|\mathcal{M}|i\rangle = -\lambda^2 \left(\frac{1}{s - m^2} + \frac{1}{u - m^2} + \frac{1}{t - m^2} \right). \tag{6.38}$$

6.2.1 Cross-section for Meson-meson Scattering

The relevant computation of the phase space integral in the center-of-mass frame has already been done in Sec. 6.1.1. It can easily be deduced from Eq. (6.30) if one factors out λ^2, the matrix element square. Overall, we have

$$\left(\frac{d\sigma}{d\Omega} \right)_{CM} = \frac{|\langle f|\mathcal{M}|i\rangle|^2}{64\pi^2 E_{CM}^2} = \frac{\lambda^4}{64\pi^2 s} \left| \frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2} \right|^2. \tag{6.39}$$

The total cross section reads (we include the 1/2 factor for identical particles):

$$\sigma = \frac{1}{2} \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \frac{d\sigma}{d\Omega}. \tag{6.40}$$

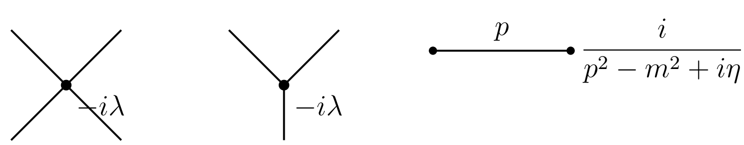
Since $E_1 = E'_1 = E$, $|\mathbf{p}_1| = |\mathbf{p}'_1|$, we have $t = (p_1 - p'_1)^2 = 2m^2 - 2p_1 \cdot p'_1 = 2m^2 - 2(E^2 - |\mathbf{p}|^2 \cos\theta) = 2(m^2 - E^2)(1 - \cos\theta)$. The integral can be rewritten with $x = \cos\theta$ as

$$\sigma = \frac{\lambda^2}{64\pi s} \int_{-1}^{1} dx \left(\frac{1}{s - m^2} + \frac{1}{3m^2 - s - 2(m^2 - E^2)(1 - x)} \right. \tag{6.41}$$
$$\left. + \frac{1}{2(m^2 - E^2)(1 - x) - m^2} \right)^2 .$$

It is possible to obtain an analytic expression for this integral, which we leave to the reader to derive.

6.3 Diagrammatic Interpretation

From the previous computations, we have seen that it is possible to provide a diagrammatic interpretation of the different terms of the S-matrix, and of its building blocks. Let us summarize them here. The building blocks read



For $\boldsymbol{\lambda\phi^4}$, these Feynman rules trivially reproduce (indeed by construction) the tree level matrix element.
For $\boldsymbol{\lambda\phi^3}$, we have one basic topology:

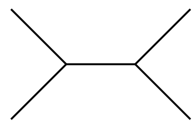


Figure 6.1: Basic topology to $\mathcal{O}(\lambda^2)$.

This topology generates three different contributions once contracted with the (fixed) initial/final states:

Tree-Level Diagrams for ϕ^3 Scattering

The three topologically distinct (once contracted with the initial/final states) diagrams that contribute at order $O(\lambda^2)$ are:

(1) s-channel

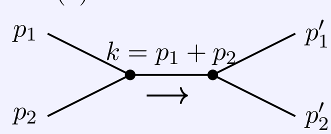


Figure 6.2: Corresponds to the term $\lambda^2/(s - m^2)$.

(2) u-channel

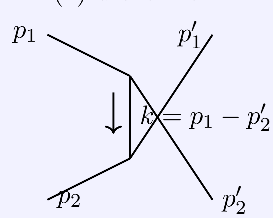


Figure 6.3: Corresponds to the term $\lambda^2/(u - m^2)$.

(3) t-channel

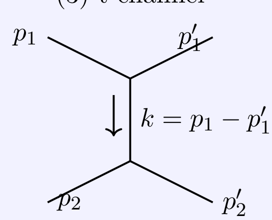


Figure 6.4: Corresponds to the term $\lambda^2/(t - m^2)$.

In these diagrams p_1 and p_2 are incoming and p_3 and p_4 outgoing. For the propagators, the arrows indicate the direction of momentum flow.

To each diagram, we assign its corresponding term in the matrix element formula:

$$(1) \to -i\lambda^2 \frac{1}{s - m^2 + i\eta}, \quad (2) \to -i\lambda^2 \frac{1}{t - m^2 + i\eta}, \quad (3) \to -i\lambda^2 \frac{1}{t - m^2 + i\eta}. \tag{6.42}$$

Then, to each vertex, we would assign a factor $(-i\lambda)$ and for each propagator a factor $\frac{i}{p^2 - m^2 + i\eta}$. In addition, we have energy-momentum conservation in each vertex. Finally, we factor out the overall energy-momentum conservation Dirac delta: $(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$.

All theories with a three-field vertex will have the basic topology shown in Fig. 6.1 to $\mathcal{O}(\lambda^2)$. Nevertheless, depending on the particle content of this three-field vertex, not all three diagrams depicted in Figs. 6.2-6.4 will necessarily show up in a physical process. For instance, in QED with relativistic fermions, charge conservation excludes some diagrams in the process $e^- e^+ \to e^- e^+$.

7 Spin-one Massless Particles: Photons

Photons are bosons characterized by having $m = 0$ and helicity $r = \pm 1$ (we combine fields with positive and negative helicity into a single field). The Fock space is formed, as usual, by states such as

$$\hat{a}^\dagger_{r_1}(\mathbf{k}_1)\ldots\hat{a}^\dagger_{r_n}(\mathbf{k}_n)|0\rangle, \tag{7.1}$$

where the $\hat{a}$'s and the $\hat{a}^\dagger$'s satisfy the commutation relations

$$[\hat{a}_r(\mathbf{k}), \hat{a}^\dagger_{r'}(\mathbf{k}')] = 2\omega_k(2\pi)^d\delta^{(d)}(\mathbf{k}-\mathbf{k}')\delta_{rr'}. \tag{7.2}$$

The Hamiltonian reads

$$H = \int d\tilde{k}\sum_r \hat{a}^\dagger_r(\mathbf{k})\hat{a}_r(\mathbf{k})\omega_k. \tag{7.3}$$

Our aim now is to obtain this Fock space from a Lagrangian formulation, as it is within the Lagrangian formulation that we can more easily introduce interactions.

The Lagrangian that describes classical electromagnetism reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.4}$$

One might try to proceed via canonical quantization. However, this approach is not viable because the Lagrangian possesses a gauge symmetry:

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu\theta\,, \tag{7.5}$$
$$\mathcal{L} \mapsto \mathcal{L}' = \mathcal{L}\,. \tag{7.6}$$

This causes some of the canonical momenta to vanish: $\Pi_\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}^\mu} \to \Pi^0 = 0$, and we cannot impose the usual commutation relation $[A_0, \Pi^0] \propto \delta$. Even if we manage to bypass this issue, additional complications arise when trying to establish canonical quantization conditions for the spatial components of A^μ. We avoid these problems by following an alternative path that does not rely on canonical quantization.

We proceed as follows. We first obtain the EoM (Maxwell equations). The Maxwell equations in the presence of external sources read

$$\partial_\mu F^{\mu\nu} = j^\mu\,, \tag{7.7}$$

in its covariant form, or

$$\nabla \cdot \mathbf{E} = \rho\,, \tag{7.8}$$

$$\nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}\,, \tag{7.9}$$

in its original form (where $j^\mu = (\rho, \mathbf{j})$). In our case $j^\mu = 0$.

If we obtain a solution to the Maxwell equations, after U(1) gauge transformation (see Eq. (7.5)), the transformed A'_μ is also a solution. We use this freedom to select a particular solution that will allow us to smoothly connect with the creation/annihilation operator picture. This is nothing but fixing the gauge. We choose the Coulomb gauge:

$$\nabla \cdot \mathbf{A}' = 0. \tag{7.10}$$

This is achieved if, for a given $\mathbf{A}$, we fix θ to be

$$\theta(t, \mathbf{x}) = \int d^d\mathbf{y} \left(\frac{-1}{4\pi|\mathbf{x} - \mathbf{y}|} \right) (\nabla_y \cdot \mathbf{A}(t, \mathbf{y})) \to \nabla \cdot \mathbf{A}' = 0. \tag{7.11}$$

In this gauge, we have

$$\nabla \cdot \mathbf{E} = 0 \Rightarrow -\Delta A^0 = 0 \Rightarrow A^0 = 0 \quad \text{if} \quad A^0(\infty) = 0. \tag{7.12}$$

Overall, the Maxwell equations simplify to

$$A^0 = 0 \qquad \text{and} \qquad \Box \mathbf{A} = 0.$$

The latter is nothing but the KG equation after setting $m = 0$. Its most general solution (with the constraint $\nabla \cdot \mathbf{A} = 0$) can be written in the following way:

$$\mathbf{A}(t, \mathbf{x}) = \int d\tilde{k} \left(\sum_{r=1,2} \boldsymbol{\epsilon}_r(\mathbf{k}) a_r(\mathbf{k}) e^{-ik\cdot x} + \sum_{r=1,2} \boldsymbol{\epsilon}^*_r(\mathbf{k}) a^\dagger_r(\mathbf{k}) e^{ik\cdot x} \right). \tag{7.13}$$

The condition $\nabla \cdot \mathbf{A} = 0$ yields

$$\mathbf{k} \cdot \boldsymbol{\epsilon}_r(\mathbf{k}) = 0. \tag{7.14}$$

We can also take the orthonormalization condition of these vectors to be $\boldsymbol{\epsilon}_r(\mathbf{k}) \cdot \boldsymbol{\epsilon}_{r'}(\mathbf{k}) = \delta_{r,r'}$. We then have the completeness relation:

$$\sum_{r=1,2} \boldsymbol{\epsilon}^i_r(\mathbf{k}) \boldsymbol{\epsilon}^j_r(\mathbf{k}) = \delta^{ij} - \frac{\mathbf{k}^i \mathbf{k}^j}{|\mathbf{k}|^2}. \tag{7.15}$$

To deduce this equation, we note that the only tensors with two indices that are rotationally invariant are δ^{ij} and $\mathbf{k}^i\mathbf{k}^j$, so $\sum_{r=1,2}\boldsymbol{\epsilon}_r^i(\mathbf{k})\boldsymbol{\epsilon}_r^j(\mathbf{k}) = A\delta^{ij} + B\mathbf{k}^i\mathbf{k}^j$. The coefficients A and B are then fixed using the normalization and transverse conditions.

So far, everything is consistent with having two massless KG fields (but with different transformation properties under Lorentz). Although we cannot apply the canonical formalism, we can write the Hamiltonian using either Eq. (1.2) or $H = P^0$, where P^0 is the generator of time translations. To proceed, we first need to obtain the canonical coordinates:

$$\begin{aligned} \pi_\mu &= \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = \frac{\partial \mathcal{L}}{\partial A^{\mu,0}} = -\frac{1}{4}\frac{\partial}{\partial A^{\mu,0}} F_{\rho\sigma}F^{\rho\sigma}. \\ &= -\frac{1}{2}F_{\rho\sigma}\frac{\partial}{\partial A^{\mu,0}}F^{\sigma\rho} = -\frac{1}{2}F_{0\mu} + \frac{1}{2}F_{\mu 0} = F_{\mu 0}\,. \end{aligned} \tag{7.16}$$

Overall, we have that ($\pi_i = \frac{\partial\mathcal{L}}{\partial(\partial_t A^i)} = -\mathbf{E}^i$)

$$\boldsymbol{\pi} = -\mathbf{E}\,. \tag{7.17}$$

Using the fact that

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \tag{7.18}$$

one obtains

$$H = \int d^d\mathbf{x}\left(\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \mathbf{E}\cdot\boldsymbol{\nabla}A_0\right), \tag{7.19}$$

where we have used the explicit expression for $\mathbf{E}$ (we keep it general for latter purposes but note that $A_0 = 0$). If we write it in terms of the (to-be) creation/annihilation operators, one obtains (see Exercise 5)

$$H = \int d\tilde{k}\sum_r (a_r^\dagger(\mathbf{k})a_r(\mathbf{k}) + a_r(\mathbf{k})a_r^\dagger(\mathbf{k}))\frac{\omega_k}{2}. \tag{7.20}$$

If we now quantize using Eq. (7.2), the Hamiltonian becomes

$$\hat{H} = \int d\tilde{k}\sum_r (\hat{a}_r^\dagger(\mathbf{k})\hat{a}_r(\mathbf{k}) + C)\omega_k. \tag{7.21}$$

Similarly to the KG case, the constant C is formally infinite but this is not a problem if we only measure energy differences.

This completes the connection with the Fock space of free photons.

Let us conclude with three further results.

Exercise

Prove that

$$[\hat{\mathbf{A}}^i(\mathbf{x}), \frac{\partial \hat{\mathbf{A}}^j}{\partial t}(\mathbf{y})] = i\left(\delta^{ij} - \frac{\boldsymbol{\nabla}^i\boldsymbol{\nabla}^j}{\boldsymbol{\nabla}^2}\right)\delta^d(\mathbf{x}-\mathbf{y}) = i\int\frac{d^d\mathbf{k}}{(2\pi)^d}\left(\delta_{ij} - \frac{\mathbf{k}^i\mathbf{k}^j}{|\mathbf{k}|^2}\right)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{7.22}$$

Therefore, $\frac{\partial \mathbf{A}^j}{\partial t}(\mathbf{y})$ is not the canonical conjugate of $\mathbf{A}^j(\mathbf{y})$.

Exercise

Prove that the photon (physical) propagator reads

$$\langle 0|T\{\hat{\mathbf{A}}^i(x), \hat{\mathbf{A}}^j(0)\}|0\rangle = \int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{i}{k^2+i\eta}\left(\delta_{ij} - \frac{\mathbf{k}^i\mathbf{k}^j}{|\mathbf{k}|^2}\right). \tag{7.23}$$

Exercise

Prove that $\hat{A}^\mu = (0, \hat{\mathbf{A}})$ does not transform as a four-vector under Lorentz transformations, i.e. it does not fulfill Eq. (5.18).

7.1 Exercises

1. Consider the Lagrangian
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A^\mu A_\mu\,.$$
Obtain the EoM. Obtain P^μ.

2. Study the continuous symmetries, and obtain the associated charges (using Noether's theorem), of the Lagrangian in the previous exercise.

3. Consider the Lagrangian
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
and work in the gauge $A_0 = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. Obtain the EoM. Obtain P^0.

4. Study the continuous symmetries and obtain the associated charges of the Lagrangian of the previous exercise using Noether's theorem.

5. Rewrite in terms of creation and annihilation operators
$$\hat{H} = \int d^d\mathbf{x}\frac{1}{2}\left(\hat{\mathbf{E}}^2 + \hat{\mathbf{B}}^2\right),$$

where $\hat{\mathbf{E}}^i = -(\partial_0 \hat{\mathbf{A}})^i$, $\mathbf{B}^i = (\boldsymbol{\nabla} \times \hat{\mathbf{A}})^i$, and

$$\hat{\mathbf{A}}^i(x) = \int d\tilde{k} \sum_{r=1,2} \left(\boldsymbol{\epsilon}_r^i(\mathbf{k}) \hat{a}_r(\mathbf{k}) e^{-ik\cdot x} + \boldsymbol{\epsilon}_r^i(\mathbf{k}) \hat{a}_r^\dagger(\mathbf{k}) e^{+ik\cdot x} \right) . \tag{7.24}$$

6. Determine whether the commutator $[A^\mu(x), A^\nu(y)]$ is zero for space-like distances.

8 Generalized Feynman Rules

In earlier chapters, we introduced some preliminary Feynman rules for scalar field theories. We now aim to systematize and extend these findings.

In this chapter, we present methods for deriving Feynman rules for vertices and propagators for a given Lagrangian. We also explain how they are assembled in a Feynman diagram. Nevertheless, we will not discuss in detail the case of relativistic fermions.

8.1 Propagators and Vertices

We first consider a Lagrangian with bosons and/or fermions with, in principle, arbitrary spin. In practice, $j = 0, 1/2, 1, 2$. We split it as follows:

$$\mathcal{L} = \mathcal{L}_0(x) + \mathcal{L}_I(x)\,, \tag{8.1}$$

with $\mathcal{L}_0$ being quadratic (free fields) and $\mathcal{L}_I$ being a polynomial (interaction term) of degree greater than or equal to three. We first discuss $\mathcal{L}_0$, from which we will obtain the propagators. It can be written as:

$$\begin{aligned}\mathcal{L}_0(x) \;=\; & \int d^4y \left(\frac{1}{2}\phi_i(x)\mathcal{P}_{ij}(x-y)\phi_j(y) + \chi_i^\dagger(x)V_{ij}(x-y)\chi_j(y)\right.\\ & \left. +\psi_i^\dagger(x)X_{ij}(x-y)\psi_j(y)\right)\,. \end{aligned} \tag{8.2}$$

$\phi_i(x)$ represents a real boson field. χ_i represents a complex boson field. ψ_j represents a fermion field. The i, j indices represent spin or internal degrees of freedom. Note that this is symbolic. For bigger spins, or with many internal quantum numbers, the fields have more than one index.

Whereas it is not standard, any free Lagrangian can be written in the form of Eq. (8.2). Once $\mathcal{L}_0$ is written in this form, one can obtain the propagator using the following method.

Propagators

To obtain the propagator, we must find the inverse of $\mathcal{P}_{ij}$:

$$\int d^4y \sum_j \mathcal{P}_{ij}(x-y)\mathcal{P}^{-1}_{jl}(y-z) = \delta_{il}\delta^{(4)}(x-z)\,. \tag{8.3}$$

This has a unique solution once we introduce the $i\eta$ prescription: $m^2 \to m^2 - i\eta$. In momentum space, it reads

$$\mathcal{P}^{-1}_{jl}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{\mathcal{P}}^{-1}_{jl}(k)\,. \tag{8.4}$$

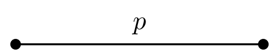
The momentum-space propagator is then defined as

Feynman Rule: Propagator

$$D_{jl}(k) = i\tilde{\mathcal{P}}^{-1}_{jl}(k). \tag{8.5}$$

Example
For the case of a real KG field, we have $\mathcal{P}(x-y) = -(\Box_x + m^2)\delta^{(4)}(x-y)$. One then easily finds that the KG propagator reads (with the replacement $m^2 \to m^2 - i\eta$)

$$D_{\rm KG}(p) = \frac{i}{p^2 - m^2 + i\eta} \tag{8.6}$$



which coincides with the Fourier transform of $i\Delta_F(x)$ (see Eq. (4.33)) and the momentum flows from left to right (though for this particular propagator it does not matter, as it is invariant under the $p \to -p$ change).

We may attempt to derive the photon propagator from Eq. (7.4) using the method described above. This leads nowhere, as the resulting propagator is not invertible. To avoid this problem, extra terms must be added to Eq. (7.4) so that the resulting $\mathcal{P}_{\mu\nu}$ becomes invertible. This procedure is known as gauge fixing in this context. There are various ways to add these extra terms to the Lagrangian, each leading to different forms of the propagator. Moreover, the resulting propagators are, in general, different from the physical propagator of the photon obtained in Eq. (7.23). We will discuss how this fits in a consistent physical theory later.

Vertices

Even if the interaction Lagrangian density is local, we can write $\mathcal{L}_I$ in the following general form:

$$\begin{aligned} \mathcal{L}_I \;&=\; \int d^4x_1 \dots d^4x_N \alpha_{i_1\dots i_m\cdots i_n\cdots i_p\cdots i_q\cdots i_N}(x; x_1, \dots, x_N) \\ &\times \psi^\dagger_{i_1}(x_1)\dots\psi_{i_m}(x_m)\cdots\phi_{i_n}(x_n)\cdots\chi^\dagger_{i_p}(x_p)\cdots\chi_{i_q}(x_q)\cdots\chi_{i_N}(x_N)\,, \end{aligned} \tag{8.7}$$

corresponding to an interaction term with $m-1$ $\Psi^\dagger$ fields, $n-1-m$ Ψ fields, $p-1-n$ ϕ fields, $q-1-p$ $\chi^\dagger$ fields, and $N-q$ χ fields.

The first step is then to rewrite the original $\mathcal{L}_I$ in the above form. This allows us to identify $\alpha_{i_1\ldots i_m\cdots i_n\cdots i_p\cdots i_q\cdots i_N}(x;x_1,\ldots,x_N)$. We are interested in the Fourier transform of this quantity:

$$\begin{aligned}&\alpha_{i_1\ldots i_m\cdots i_n\cdots i_p\cdots i_q\cdots i_N}(x_1,\ldots,x_N)\\&=\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_N}{(2\pi)^4}e^{-ik_1\cdot(x-x_1)}\cdots e^{-ik_N\cdot(x-x_N)}\tilde{\alpha}_{i_1\ldots i_m\cdots i_n\cdots i_p\cdots i_q\cdots i_N}(k_1,\ldots,k_N).\end{aligned}\tag{8.8}$$

Note that the derivatives become multiplicative factors after applying the Fourier transform.

Important: The minus signs in the exponents indicate that the momentum flow in the Feynman rules is always incoming. This is something that should be taken into account when drawing and computing diagrams.

The Feynman rule associated with the vertex reads as follows:

Feynman Rule: Vertex

$$I=i\sum_{\substack{\mathcal{P}erm\\(1,m-1)}}\sum_{\substack{\mathcal{P}erm\\(m,n-1)}}\sum_{\substack{\mathcal{P}erm\\(n,p-1)}}\sum_{\substack{\mathcal{P}erm\\(p,q-1)}}\sum_{\substack{\mathcal{P}erm\\(q,N)}}(-1)^P\tilde{\alpha}_{i_1\ldots i_m\cdots i_n\cdots i_p\cdots i_q\cdots i_N}(k_1,\ldots,k_N)\tag{8.9}$$

$(-1)^P$ stands for the fact that a (-1) factor has to be introduced each time two fermion fields are permuted.
Note also that, when one performs permutations between the quantum numbers of identical particles, all quantum numbers should be interchanged: both labels *and* momenta.

Example. $\lambda\phi^4$.

$$\begin{aligned}&\mathcal{L}(x)=-\frac{\lambda}{4!}\phi^4(x)=\\&=-\frac{\lambda}{4!}\int d^4x_1d^4x_2d^4x_3d^4x_4\delta^{(4)}(x-x_1)\delta^{(4)}(x-x_2)\delta^{(4)}(x-x_3)\delta^{(4)}(x-x_4)\\&\times\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\end{aligned}\tag{8.10}$$

so

$$\alpha(x;x_1,x_2,x_3,x_4)=-\frac{\lambda}{4!}\prod_{i=1}^{4}\delta^{(4)}(x-x_i)\implies\tilde{\alpha}=-\frac{\lambda}{4!}\tag{8.11}$$

and the Feynman rule for the vertex reads

$$I = i \sum_{\mathcal{P}erm(4)} \left(-\frac{\lambda}{4!}\right) = -i\lambda \,. \tag{8.12}$$

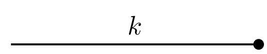
Final Feynman Rules

- Vertex: $I \times (2\pi)^4 \delta^{(4)}(\sum k_i)$. The Dirac delta enforces momentum conservation. Remember that all momenta are incoming.
- Propagator (always internal): $\int \frac{d^D k}{(2\pi)^D} D_{\alpha\beta}(k)$
- In tree level computations, the integrals and Dirac deltas compensate each other up to an overall energy-momentum conservation Dirac delta for the external particles that factors out. In such cases one could simplify the above Feynman rules and work directly with I's and D's, imposing energy-momentum conservation in each vertex, and multiplying the final result by an overall energy-momentum conservation Dirac delta for the external particles. However, this would not be correct in the general case, in particular, with loops.

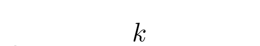
8.2 Feynman Rules for External Particles

- **Real Scalar:** We have already worked out the case of the real scalar field. The relevant computation was

$$\hat{\phi}^{(+)}(x)\hat{a}^\dagger(\mathbf{k})|0\rangle = 1 \cdot e^{-ik\cdot x}|0\rangle \tag{8.13}$$



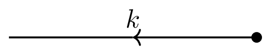
The exponential gets embedded in the energy-momentum conservation Dirac delta (note the minus sign indicating they are incoming particles and that $k^0 = w_k$). The factor **1** is the Feynman rule for both incoming and outgoing particles with momentum k. The latter is drawn in the following way



- **Complex Scalar:** Particle and antiparticle are different states in this case. Which one is designated the particle and which one the antiparticle is a matter of convention. We have that $\phi \sim a + b^\dagger$. We again have that

$$\hat{\phi}^{(+)}(x)\hat{a}^\dagger(\mathbf{k})|0\rangle = 1 \cdot e^{-ik\cdot x}|0\rangle \tag{8.14}$$

$$\hat{\phi}^{\dagger(+)}(x)\hat{b}^{\dagger}(\mathbf{k})|0\rangle = 1 \cdot e^{-ik\cdot x}|0\rangle \tag{8.15}$$



and similarly for the outgoing particles:

Overall, the Feynman rule for external complex scalars is a factor **1** for incoming/outgoing particles and antiparticles with momentum k. Arrows indicate the charge flow and distinguish particles from antiparticles.

We can easily obtain the generalization to particles with general spin.

- **Massless Spin-1 (Photon):** For a (physical[1]) photon, we have

$$\hat{A}_{\mu}^{(+)}(x) = \int d\tilde{k} \sum_{\lambda} \epsilon_{\mu}(\mathbf{k},\lambda)\hat{a}(\mathbf{k},\lambda)e^{-ik\cdot x}\,, \tag{8.16}$$

  and

$$\hat{A}_{\mu}^{(+)}(x)\hat{a}^{\dagger}(\mathbf{k},\lambda)|0\rangle = \epsilon_{\mu}(\mathbf{k},\lambda)e^{-ik\cdot x}|0\rangle\,. \tag{8.17}$$

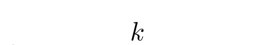
  Similarly,

$$\langle 0|\hat{a}(\mathbf{k},\lambda)\hat{A}_{\mu}^{(-)}(x) = \langle 0|\epsilon_{\mu}^{*}(\mathbf{k},\lambda)e^{ik\cdot x} \tag{8.18}$$

  Overall, the Feynman rules read

  - Incoming: $\epsilon_{\mu}(\mathbf{k},\lambda)$

  - Outgoing: $\epsilon_{\mu}^{*}(\mathbf{k},\lambda)$



[1]Is there any other kind of photon?

- **NR spin-$\frac{1}{2}$ Fermions:** The NR fermion field is a vector of dimension two. We write it as follows

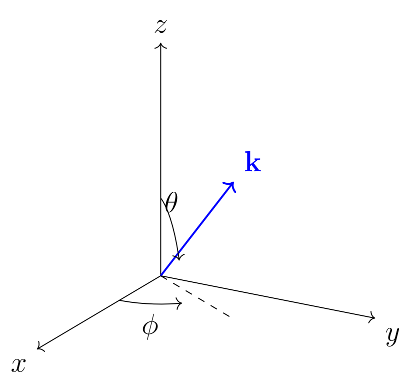
$$\hat{\varphi}(x) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_{r=\pm} \alpha(\hat{k},r)\hat{b}_r(\mathbf{k})e^{-ik\cdot x}\,, \tag{8.19}$$

$$\hat{\chi}_c(x) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_{r=\pm} \alpha_c(\hat{k},r)\hat{d}_r(\mathbf{k})e^{-ik\cdot x}\,. \tag{8.20}$$

Here, we will consider two possibilities. One is more suitable for the NR limit, whereas the other allows for a smoother transition to the relativistic case. The difference will come from how we define α. One option is to quantize along the axis of motion of the NR particle:

$$\alpha(\hat{k},+) = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \qquad \alpha(\hat{k},-) = \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{8.21}$$

where the vector $\mathbf{k}$ is represented in the following way



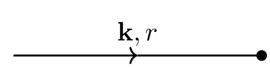
The other option is to make it independent of the momentum of the particle:

$$\alpha(\hat{k},+) \equiv \alpha(+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \alpha(\hat{k},-) \equiv \alpha(-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8.22}$$

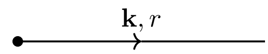
This is convenient in the NR limit.

Regardless of the basis, the Feynman rules always have the following form.

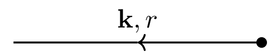
  - Incoming fermion: $\alpha(\hat{k},r)$



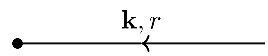
– Outgoing fermion: $\alpha^*(\hat{k}, r)$



– Incoming antifermion: $\alpha_c(\hat{k}, r)$



– Outgoing antifermion: $\alpha_c^*(\hat{k}, r)$



For convenience, we put arrows to distinguish particles from antiparticles, although we could treat them separately.

8.3 Matrix Elements

The amplitude $i\mathcal{M}$ is constructed by multiplying these factors together:

$$i\mathcal{M} = (\text{external particles}) \times (\text{Vertices}) \times (\text{Propagators})\,, \tag{8.23}$$

and integrating over any undetermined internal momenta. The indices of the vertices are contracted with those of the propagators and external particles to which they are connected.

The above discussion suffices for the purposes of this book. Additional subtleties that may arise at loop level are not discussed here.

8.4 Exercises

1. Compute the Feynman propagator of the photon field from the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2a}(\partial_\mu A^\mu)^2 \tag{8.24}$$

in momentum space using the generalized Feynman rule methods.

2. Compute the Feynman propagator of the Proca field in momentum space using the generalized Feynman rule methods.

3. Derive the Feynman rules for the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2 - \frac{\lambda}{4}\phi_1^2\phi_2^2 \, .$$

4. Obtain the Feynman rule for the following vertex of scalar fields

$$\delta\mathcal{L} = (\partial_\mu\phi^2)(\partial^\mu\phi^2) \, .$$

5. Derive the Feynman rule for the vertex $\frac{\lambda}{3!}\phi^3$.
6. Discuss whether the following processes (H is a scalar particle): $HHH \to H$ and vacuum $\to HHHH$, are physically possible within $\lambda\phi^n$ theories.
7. Derive the Feynman rules for $\lambda\phi^2(\phi^*)^2$ and $\lambda\phi_1^2\phi_2^3$ theories.
8. Derive the Feynman rules of the following interaction vertex

$$\delta\mathcal{L} = \frac{\alpha N_c}{24\pi F_\pi}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\phi \, ,$$

where $\phi(x)$ is a real scalar field (for instance π^0: the chargeless pion field) and A_μ is the field of the photon.

9. Using the vertex of the previous exercise (with $N_c = 3$ and $F_\pi \simeq 92$ MeV) compute the decay width of the process $\pi^0 \to \gamma\gamma$. Consult the PDG and compare with the experimental number.

9 QFTs with Photons

The interaction between massive particles and a single massless spin-one particle is what is known as Quantum electrodynamics (QED). In most standard QFT textbooks, the massive particles are relativistic spin-$\frac{1}{2}$ fermions. In these introductory notes, we do not study relativistic spin-$\frac{1}{2}$ fermions. Therefore, we have to take a different route. We will consider the interaction of the photon with either massive relativistic charged scalars or with NR particles of arbitrary spin. This will be more than enough for the purposes of this book. Nevertheless, this choice will also force us to make some modifications in some derivations.

As the name QED suggests, we are dealing with a quantum theory. There are two main routes to construct interacting QFTs involving photons:

1) One is to first pose the problem in a classical setup. We consider the Lagrangian of free matter fields and enforce gauge invariance on it. The solution to this problem is the introduction of a four-vector (under Lorentz transformations) field, which transforms inhomogeneously under gauge transformations. This is the traditional route to derive the QED Lagrangian and the one we will follow for the matter fields we consider in this book. One then tries to quantize the resulting Lagrangian. This is complicated in general. We will only work out the NR case in detail and state the solution for the scalar relativistic one.

2) The other path is to consider the most general interacting theory of massless spin-one particles (photons) with matter that complies with Poincaré invariance. As appealing as this approach is, we will not discuss it here, as it would require additional preparatory material. The interested reader can find a discussion in [4].

9.1 Scalar QED

The Lagrangian of scalar QED can be derived from the gauge principle. We start with the Lagrangian density for a free, charged, scalar particle:

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2|\phi|^2 \,. \tag{9.1}$$

This Lagrangian density is invariant under a **global** U(1) transformation, $\phi(x) \mapsto e^{-i\alpha e}\phi(x)$, which leads to the conserved Noether's current (see Eq. (4.62)):

$$j_\mu = ie[\phi^*(\partial_\mu \phi) - (\partial_\mu \phi^*)\phi] \equiv ie\phi^* \overleftrightarrow{\partial_\mu} \phi \,. \tag{9.2}$$

The above symmetry corresponds to performing the same "rotation" of the field at every point in space-time—a "rigid" transformation. A natural question arises: Why not consider a more general type of rotation? See the discussion in [9]. This would correspond to allowing the field redefinition to vary from point to point in space-time (a "local" transformation). If we require invariance under a **local** transformation: $\alpha \to \alpha(x)$, the Lagrangian is no longer invariant, nor is the difference a total derivative:[1]

$$\mathcal{L} \to \mathcal{L} + (\partial_\mu \alpha) j^\mu \,. \tag{9.3}$$

To enforce gauge invariance, we replace the ordinary derivative ∂_μ with the **covariant derivative** D_μ:

$$\partial_\mu \qquad \to \qquad D_\mu = \partial_\mu + ieA_\mu \,. \tag{9.4}$$

This introduces a new vector field: A_μ (the photon field), which must transform under gauge transformations as

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x) \,. \tag{9.5}$$

This ensures that

$$D_\mu \phi \to e^{-i\alpha(x)e} D_\mu \phi(x) \tag{9.6}$$

transforms homogeneously under the U(1) gauge transformations, unlike $\partial_\mu \phi(x)$. Therefore, we have that

$$(D_\mu \phi)^* (D^\mu \phi) - m^2 |\phi|^2 \tag{9.7}$$

is invariant under U(1) gauge transformations. This is not the whole history, however, as this invariance has been achieved at the cost of introducing an additional field, A_μ. Therefore, we must now look for additional operators made of A_μ's (or with A_μ content) with good transformation properties. In this respect, the following identity proves useful:

$$[D_\mu, D_\nu]\phi = [\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu]\phi = ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\phi = ieF_{\mu\nu}\phi.$$

We have then that $[D_\mu, D_\nu]$ is *gauge invariant* and thus $F_{\mu\nu}$ too.[2] This can easily be generalized to non-abelian gauge theories where it turns out to be more useful.

[1] One might be tempted to perform an integration by parts so that the difference becomes $\propto \alpha(x)\partial_\mu j^\mu$. However, note that Noether's theorem requires the variation to be, at most, a total derivative *before* applying the EoM so that, even if j^μ is a conserved current, we cannot set $\partial_\mu j^\mu = 0$.

[2] Something that we already knew from Chapter 7.

The complete gauge-invariant Lagrangian for scalar QED must include a kinetic term for the new field A_μ. The only term that is simultaneously U(1) gauge-invariant, Poincaré invariant, and quadratic in the photon field is $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ (the normalization is fixed, for convenience, as in Eq. (7.4)). The $U(1)$ gauge symmetry forbids an $A_\mu A^\mu$ term so the photon is massless.

Adding the photon kinetic term to Eq. (9.7), the Lagrangian reads

$$\boxed{\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - m^2|\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.} \tag{9.8}$$

If we neglect a possible $\lambda|\phi|^4$ term, this is the only possible Lagrangian under the conditions of having a single charged scalar field, U(1) gauge symmetry, Poincaré invariance, C, P, and T symmetry, and perturbative renormalizability. Perturbative renormalizability means that the ultraviolet divergences generated in perturbation theory can be reabsorbed into redefinitions of the paramaters of the Lagrangian. In this book we do not delve into the issue of renormalizability of QFTs. We will only mention that perturbative renormalizability imposes a necessary condition on the allowed operators that can appear in the Lagrangian: The only allowed operators are those with field content $[O] = M^{D_O}$ with $D_O \leq 4$. This means that we can only have superrenormalizable operators (with $D_O < 4$) or marginal (renormalizable) operators (with $D_O = 4$). Non-renormalizable operators (with $D_O > 4$) are not allowed. An example of such an operator is

$$(D_\mu\phi)^*(D^\mu\phi)|\phi|^2\,,$$

which has dimension 6, since $[\phi] = M^1$ and $[A] = M^1$. Note that these non-renormalizable operators get multiplied by dimensionful constants with negative dimensions in the Lagrangian. From an effective field theory point of view these terms are then less important at low momentum transfer.

Exercise

Show that the equations of motion derived from the Lagrangian (9.8) are

$$\partial_\mu F^{\mu\nu} = J^\nu \quad ; \quad \left[(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu) + m^2\right]\phi = 0$$

where

$$J^\nu = j^\nu - 2e^2 A^\nu \phi^*\phi \quad ; \quad j^\nu = ie\phi^* \overleftrightarrow{\partial^\nu}\phi\,,$$

and J^ν is the total electromagnetic current associated with the global U(1) symmetry of Eq. (9.8).

In the above discussion, we have considered a single charged particle. However, we could consider a more general scenario involving different particles, each potentially carrying a different charge. Additionally, we have focused solely on deriving the classical Lagrangian of scalar QED. It is also possible to write down the classical Hamiltonian, which is left as an exercise for the reader. However, we do not address its quantization in this book. Instead, we will carry out the quantization for the NR theory in the next section. This will prove sufficient for the purposes of these introductory notes.

9.2 NRQED

We begin by considering a NR particle of arbitrary spin, which we describe using the field φ. This particle may be either a boson or a fermion, representing a matter field. We will typically consider either scalars or spin-$\frac{1}{2}$ particles. In the first case, the field has a single component, whereas in the second, the field is a vector of dimension two, i.e., it has two components. Most of our discussion will be spin-independent, since spin-dependent effects are typically suppressed by powers of the mass of the NR particle. Therefore, we can omit the index as the interactions will be diagonal in spin space.[3]

The Lagrangian of a NR particle at leading nonvanishing order in the velocity expansion reads

$$\mathcal{L} = \varphi^\dagger \left(i\partial_0 + \frac{\boldsymbol{\nabla}^2}{2m} \right) \varphi \,. \tag{9.9}$$

This Lagrangian is invariant under U(1) global symmetry. As in the previous section, we enforce U(1) local symmetry. Again, the solution is to transform normal derivatives into covariant derivatives and add the photon Lagrangian term

$$\mathcal{L} = \varphi^\dagger(t, \mathbf{x}) \left\{ iD_0 + \frac{\mathbf{D}^2}{2m} \right\} \varphi(t, \mathbf{x}) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{9.10}$$

with (see Eq. (1.1))

$$iD_0 = i\partial_0 - eA_0, \qquad i\mathbf{D} = i\boldsymbol{\nabla} + e\mathbf{A} \,. \tag{9.11}$$

This Lagrangian is perturbatively nonrenormalizable from the start. Therefore, the usual restriction of requiring perturbative renormalizability does not apply, and one could, in principle, add further terms to the Lagrangian while still preserving gauge invariance. Such terms would be subleading in the NR limit. Therefore, we neglect them here, as one would do in effective field theories at low energies.

[3]When this is not the case, we will state it explicitly.

Exercise

Show that the EoM from the Lagrangian (9.10) are:

$$\left(iD_0 + \frac{1}{2m}\mathbf{D}^2\right)\varphi = 0\,, \tag{9.12}$$

$$\partial_\mu F^{\mu 0} = e\varphi^\dagger\varphi\,, \quad \partial_\mu F^{\mu i} = -\frac{ie}{2m}(\varphi^\dagger(\boldsymbol{\nabla}^i\varphi) - (\boldsymbol{\nabla}^i\psi)^\dagger\varphi)\,. \tag{9.13}$$

Eqs. (9.13) are nothing but the Maxwell equations in the presence of (NR) sources (see Eqs. (7.8) and (7.9)). Note that, without matter ($\rho = 0$), we had $A_0 = 0$ (see Eq. (7.12)). Since, we now have charged matter fields ($\rho \neq 0$),

$$\boldsymbol{\nabla}\cdot\mathbf{E} = -\boldsymbol{\nabla}\cdot\dot{\mathbf{A}} - \boldsymbol{\nabla}^2 A_0 = e\rho\,,$$

which simplifies to ($\boldsymbol{\nabla}\cdot\mathbf{A} = 0$)

$$\boldsymbol{\nabla}^2 A_0 = -e\varphi^\dagger\varphi\,, \tag{9.14}$$

and we can solve this equation (note that there is no evolution in time):

$$A_0(t,\mathbf{x}) = \frac{e}{4\pi}\int d^d\mathbf{x}'\,\frac{\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{9.15}$$

where

$$\rho = j^0(x) = \varphi^\dagger(x)\varphi(x). \tag{9.16}$$

Overall, the outcome of this computation is that A_0 is not an independent field and that it is non-zero.

9.2.1 Quantization

Let us now turn to the Hamiltonian formulation. Our goal is to express the Hamiltonian in terms of creation and annihilation operators, and to decompose it into the free and interaction terms. To this end, we adapt the derivation presented in [8] for relativistic spin-$\frac{1}{2}$ particles to our particular NR setup.

Using the canonical momenta:

$$\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{A}}^i} = -\mathbf{E}^i = \boldsymbol{\nabla}^i A_0 + \dot{\mathbf{A}}^i \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = i\varphi^\dagger\,, \tag{9.17}$$

the Hamiltonian reads

$$\begin{aligned} H \;=\; & \int d^d\mathbf{x}\left\{\varphi^\dagger\left(-\frac{1}{2m}\right)\boldsymbol{\nabla}^2\varphi + \frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2) + \mathbf{E}\cdot(\boldsymbol{\nabla}A_0) + e\varphi^\dagger\varphi A_0 \right.\\ & \left. + \frac{ie\mathbf{A}}{2m}[\varphi^\dagger(\boldsymbol{\nabla}\varphi) - (\boldsymbol{\nabla}\varphi^\dagger)\varphi] + \frac{e^2}{2m}\mathbf{A}^2\varphi^\dagger\varphi\right\}\,. \end{aligned} \tag{9.18}$$

This expression does not yet have the free and interaction terms separated, because A_0 and $\mathbf{E}$ depend on the fields $\varphi, \varphi^\dagger$. We need to eliminate A_0. Let us first note that

$$\int d^d\mathbf{x}\mathbf{E}\cdot\boldsymbol{\nabla}A_0 = -\int d^d\mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{E})A_0 = -e\int d^d\mathbf{x}\rho A_0\,. \tag{9.19}$$

We now split $\mathbf{E}$ into its transverse and longitudinal components: $\mathbf{E} = \mathbf{E}_T + \mathbf{E}_L$ where, by definition, the transverse part satisfies $\boldsymbol{\nabla}\cdot\mathbf{E}_T = 0$. Then, $\mathbf{E}_T = -\dot{\mathbf{A}}$ and $\mathbf{E}_L = -\boldsymbol{\nabla}A_0$. Thus, we have

$$\int d^d\mathbf{x}\mathbf{E}^2 = \int d^d\mathbf{x}(\mathbf{E}_T^2 + 2\mathbf{E}_T\cdot\mathbf{E}_L + \mathbf{E}_L^2). \tag{9.20}$$

We now study the terms proportional to the longitudinal electric field. We first have

$$\int d^d\mathbf{x}2\mathbf{E}_T\cdot\mathbf{E}_L = 2\int d^d\mathbf{x}(-\boldsymbol{\nabla}A_0)\cdot(-\dot{\mathbf{A}}) = -2\int d^d\mathbf{x}A_0(\boldsymbol{\nabla}\cdot\dot{\mathbf{A}}) = 0. \tag{9.21}$$

For the other term, we obtain[4]

$$\begin{aligned}\int d^d\mathbf{x}\mathbf{E}_L^2 &= \int d^d\mathbf{x}(\boldsymbol{\nabla}A_0)\cdot(\boldsymbol{\nabla}A_0) = -\int d^d\mathbf{x}A_0(\boldsymbol{\nabla}^2A_0) = e\int d^d\mathbf{x}A_0\rho \\ &= \frac{e^2}{4\pi}\int dxdy\frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,. \end{aligned}\tag{9.23}$$

In the end, the Hamiltonian reads

$$H = H_0 + H_I\,, \tag{9.24}$$

where

$$H_0 = \int d^d\mathbf{x}\left\{\varphi^\dagger\left(\frac{-\boldsymbol{\nabla}^2}{2m}\right)\varphi + \frac{1}{2}(\mathbf{E}_T^2 + \mathbf{B}^2)\right\}\,, \tag{9.25}$$

and

$$\begin{aligned}H_I &= \int d^d\mathbf{x}\left\{\frac{ie\mathbf{A}}{2m}(\varphi^\dagger(\boldsymbol{\nabla}\varphi) - (\boldsymbol{\nabla}\varphi^\dagger)\varphi) + \frac{e^2}{2m}\mathbf{A}^2\varphi^\dagger\varphi\right\} \\ &\quad + \frac{e^2}{8\pi}\int d^d\mathbf{x}d^d\mathbf{y}\frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,. \end{aligned}\tag{9.26}$$

[4]It is useful to recall the following identities:

$$\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2} = \frac{1}{4\pi}\frac{1}{r} \equiv -\frac{1}{\boldsymbol{\nabla}^2}\delta(\mathbf{r}) \tag{9.22}$$

This expression can be quantized directly. $\hat{H}_0$ is the Hamiltonian of free NR particles and free photons. $\hat{H}_I$ is the interaction term (it is a polynomial of creation and annihilation operators with degree greater than or equal to three). It is made out of two terms. The first is a standard local term (albeit including NR fields) for which we can directly obtain the Feynman rules using the methods discussed in Chapter 8. The second is non-local in space. It resembles a Coulomb-like interaction between NR charged particles. For this term, a straightforward application of the Feynman rules presented in Chapter 8 is more difficult (though possible). We will discuss the handling of this term in the following section. In any case, it should be stressed that the appearance of this non-local (in space) interaction term is a general feature of QED, regardless of whether one uses relativistic or NR matter fields for the derivation (see [8]).

9.2.2 S-matrix and (effective) Photon Propagator

In Eq. (7.23), the propagator of the (physical) photon was obtained. This generates the following Feynman rule in momentum space:

$$D^{ij}_{\mathbf{A}}(p) = \frac{i}{k^2 + i\eta}\left(\delta^{ij} - \frac{\mathbf{k}^i\mathbf{k}^j}{|\mathbf{k}|^2}\right) \tag{9.27}$$

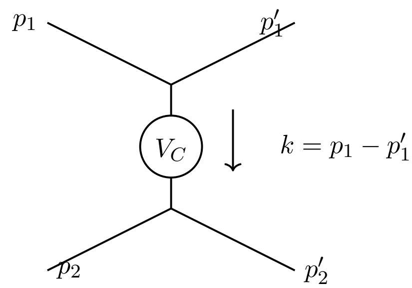
This expression does not agree with the photon propagator one obtains from the Lagrangian in Eq. (8.24). Let us see what is going on.

In the previous section, a non-local (in space) interaction term

$$\delta\hat{H}_I = \frac{e^2}{8\pi}\int d^d\mathbf{x}d^d\mathbf{y}\frac{\hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \tag{9.28}$$

appeared in $\hat{H}_I$. This term resembles very much a Coulomb potential interaction. We cannot diagrammatically represent such a term as a local vertex. We use the following diagrammatic representation instead[5]

[5] In this chapter, we do not draw arrows for charged particles to simplify the discussion. We will carefully discuss how to put arrows in the next chapter.



The vertical line is really a vertical line, since the time is the same both in the up and down vertices. In momentum space, this manifests as the absence of any k^0 dependence.

If we insist on writing all interactions in terms of local vertices, we must rewrite the non-local interaction as the propagator of a fictitious particle (somewhat reintroducing A^0). This particle is non-physical, as it does not propagate in time. This non-physical component, A_0, mediates the instantaneous Coulomb interaction if we introduce the following local interaction vertex:

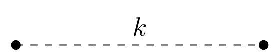
$$\delta \hat{H}_I = e\hat{A}_0 \hat{\varphi}^\dagger \hat{\varphi}\,, \tag{9.29}$$

and use the following *definition* for the A_0 propagator:

$$\langle 0|T\{\hat{A}_0(x)\hat{A}_0(0)\}|0\rangle \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{\mathbf{k}^2} = \delta(x^0)\frac{i}{|\mathbf{x}|}\frac{1}{4\pi}\,. \tag{9.30}$$

In momentum space, we associate the following Feynman rule to this propagator:

$$\frac{i}{\mathbf{k}^2} \tag{9.31}$$



The above procedure succeeds in eliminating the nonlocal interaction term, which is generated now by the following diagram:

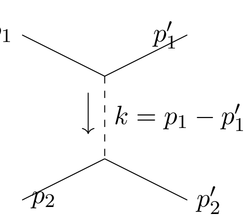


Figure 9.1: Up to a factor, it yields a contribution $\frac{i}{\mathbf{k}^2}$ to the S-matrix.

Thus, we have two alternative ways to handle the non-local interaction term in the S-matrix. One is to use Eq. (9.28). The other is to use the A_0 propagator given in Eq. (9.30) along with the Feynman rule generated by the vertex in Eq. (9.29) with the qualification that A_0 can never be an asymptotic particle. Indeed, we cannot emphasize enough that Eq. (9.30) is wrong in all possible ways: There is no creation/annihilation operator nor a vacuum state associated with A_0. We write it in this way purely for convenience. Indeed, an additional bonus of this approach is that we can combine Eq. (9.30) and Eq. (7.23) into a single, formally covariant, object—the propagator:

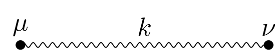
$$\langle 0|T\{\hat{A}_\mu(x)\hat{A}_\nu(0)\}|0\rangle \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{iP_{\mu\nu}}{k^2+i\eta}\,, \tag{9.32}$$

where[6]

$$P^{\mu\nu} = -g^{\mu\nu} - \frac{k^\mu k^\nu - (k\cdot n)n^\mu k^\nu - (k\cdot n)n^\nu k^\mu}{\mathbf{k}^2} \quad \text{with } n^\mu = (1,\mathbf{0}). \tag{9.33}$$

One can easily check that for label indices (i,j) one recovers the physical propagator, i.e. Eq. (9.27), for label indices $(0,i)$ one gets zero, and for label indices (0,0), we recover Eq. (9.31). We then represent this propagator in the following way:

$$D_{\mu\nu}(p) = \frac{iP_{\mu\nu}}{k^2+i\eta} \tag{9.34}$$



Observation. We used wavy lines for the external photons in Sec. 8.2. We could equally well have used the zig-zag symbol, as only physical photons appear as asymptotic states.

In NR theories, the longitudinal and transverse components are naturally separated in the interaction terms. Therefore, one might wonder about the necessity of putting them together. Nevertheless, combining them in a single propagator is almost a must in a relativistic setup. On top of that, the convenience of combining both contributions together gets reinforced by the gauge invariance of the S-matrix, which mixes longitudinal and transverse components. We discuss this issue in the next section. Finally, as we have already seen, an extra bonus comes from the fact that the non-local term can be eliminated by adding a local interaction term depending on this fictitious A_0 field.

[6]Note that $P^{\mu\nu}$ reduces to P^{ij} when computed on-shell since, for P^{00}, we obtain $-g^{00} + \frac{k^0k^0}{\mathbf{k}^2} = \frac{k^2}{\mathbf{k}^2}$ which is zero on-shell.

9.2.3 Gauge Invariance of the S-matrix

The attentive reader (and dedicated student) who has completed Exercise 1 in Sec. 8.4 may have noticed that we have not yet solved the initial problem posed below Eq. (9.27). Even if we now take Eq. (9.34)—the Fourier transform of Eq. (9.32)—as the effective photon propagator, this expression is different (in general) from the one obtained by applying generalized Feynman rule methods to the Lagrangian in Eq. (8.24).

This problem is resolved by using the identities that the S-matrix satisfies due to gauge symmetry. This can be stated as a theorem (see the discussion on p. 450 of [4]).

Theorem: Gauge invariance of the S-matrix

S-matrix elements are unaffected by changing any photon propagator $D_{\mu\nu}(q)$ in momentum space by the following transformation

$$D_{\mu\nu}(q) \to D_{\mu\nu}(q) + \alpha_\mu q_\nu + \beta_\nu q_\mu \,, \tag{9.35}$$

and/or the polarizations by the following transformation

$$\epsilon_\mu(k,\lambda) \to \epsilon_\mu(k,\lambda) + c k_\mu \,, \tag{9.36}$$

where α_μ, β_ν, c are not necessarily constants (they could be functions of q/k). Here, q_μ does not need to be on-shell ($q_0 \neq |\mathbf{q}|$), but k_μ does because it corresponds to a physical photon ($k_0 = |\vec{k}|$).
An important qualification to this theorem is that such transformations must be applied to the full S-matrix, not to individual diagrams.

We will not prove this theorem but only show that it holds for the quantities we compute.

It is easy to see that this theorem solves our initial problem. Propagators that differ by terms proportional to q_μ and/or q_ν are equally legitimate. If we compare Eq. (9.32) with the result obtained in Exercise 1 in Sec. 8.4, the difference between the propagators is indeed proportional to q_μ and/or q_ν. In fact, we could use this freedom to choose propagators with a simpler or more convenient form. One example is the Feynman gauge, where the photon propagator takes the form

$$\langle 0|T\{\hat{A}_\mu(x)\hat{A}_\nu(0)\}|0\rangle|_{FG} = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2+i\eta}(-g_{\mu\nu}) = g_{\mu\nu}\frac{1}{4i\pi^2}\frac{1}{x^2-i\eta} \tag{9.37}$$

i.e., we set $P^{\mu\nu}(k) = -g^{\mu\nu}$, and still physics remains unchanged. Recall that this statement applies to the full S-matrix, individual diagrams may differ.

This closes the loop. With this final qualification, we can use the generalized Feynman rules for this NR QFT with the assignment

$$\mathcal{H}_I = -\mathcal{L}_I \,, \tag{9.38}$$

and use any photon propagator obtained from the generalized Feynman rule methods with an arbitrary gauge fixing term as discussed in Chapter 8.

Note also that the theorem above shows that the S-matrix element is invariant under changes of the polarization of the external particles. The combined transformations of the propagator and the polarizations of the external particles are simply the gauge transformations of the A_μ field.

For relativistic spin-$\frac{1}{2}$ particles, the construction is similar and can be found in [8]. If the dependence of the Lagrangian on A_0 is more complicated, similar results can be obtained but the derivation is more cumbersome. This applies, in particular, to scalar QED. Therefore, we can use the generalized Feynman rules in this case too (at least for the purposes of this book), again using the assignment:

$$\mathcal{H}_I = -\mathcal{L}_I \,, \tag{9.39}$$

and any photon propagator obtained after gauge fixing. This is how we will proceed for scalar QED in the next chapter.[7]

[7]It also remains to be explicitly shown that the resulting S-operator is a scalar under Poincaré transformations. This is deferred to a more advanced textbook.

10 Scalar QED. Applications

In this chapter, we aim to compute some observables in scalar QED, the theory presented in Sec. 9.1. We will see how relativity in QFTs relates different physical processes and verify gauge invariance. First, we need the Feynman rules, which we will derive in the next section. We will use these examples to highlight the subtleties of the Feynman rules for charged particles.

10.1 Feynman Rules of Scalar QED

The Lagrangian is written in Eq. (9.8). Propagators have already been computed. The propagator of a charged scalar field is equal to the propagator of a real scalar field. The photon propagator has been discussed in Sec. 9.2.2 and Sec. 9.2.3, where it was emphasized that one can use any gauge for its computation.

We now consider the interaction terms. We split the interaction Lagrangian density into two terms:

$$\delta\mathcal{L} = \underbrace{ie(\partial_\mu\phi^*)A^\mu\phi - ie\phi^* A^\mu(\partial_\mu\phi)}_{\delta\mathcal{L}^{(1)}} + \underbrace{e^2 A_\mu A^\mu \phi^*\phi}_{\delta\mathcal{L}^{(2)}} \,. \tag{10.1}$$

For the first term, we have

$$\begin{aligned}\delta\mathcal{L}^{(1)} = &\int d^4x_1 d^4x_2 d^4x_3 A^{\mu_1}(x_1)\phi^*(x_2)\phi(x_3) \\ &\underbrace{(ie)\delta^{(4)}(x-x_1)[(\partial^{(x)}_{\mu_1}\delta^{(4)}(x-x_2))\delta^{(4)}(x-x_3) - \delta^{(4)}(x-x_2)(\partial^{(x)}_{\mu_1}\delta^{(4)}(x-x_3))]}_{\alpha_{\mu_1}(x;x_1,x_2,x_3)} \,.\end{aligned} \tag{10.2}$$

Changing to momentum space

$$\delta^{(4)}(x-x_i) = \int \frac{d^4k_i}{(2\pi)^4} e^{\overset{\text{incoming}}{-ik_i\cdot(x-x_i)}}, \quad \partial^x_\mu\delta^{(4)}(x-x_i) = \int \frac{d^4k_i}{(2\pi)^4}(-ik_{i\mu})e^{-ik_i\cdot(x-x_i)} \,, \tag{10.3}$$

we can obtain the Fourier transform of α_μ and the Feynman rule for the vertex. Since all fields are different, there are no combinatorial factors involved. Overall, we obtain

$$I = i\tilde{\alpha}_\mu = i(+ie)\cdot 1\cdot\{(-ik_{2\mu})\cdot 1 - 1\cdot(-ik_{3\mu})\} = ie(k_{2\mu} - k_{3\mu}) \,. \tag{10.4}$$

To give the diagrammatic representation of this Feynman rule, we also need to assign arrows to the legs of the vertex. This is a delicate issue. The internal arrows refer to charge, while the external arrows represent momenta. Charge is conserved, but we need a convention to distinguish between what we call "particle" and what we call "antiparticle". If $\phi \sim \hat{a}+\hat{b}^{\dagger}$, we assign $\hat{a}$ to the particle and $\hat{b}$ to the antiparticle. It should be emphasized that, when asked about providing Feynman rules, we should give Eq. (10.4) along with the corresponding diagram (with arrows, and indicating which operator corresponds to the particle), which in this case reads

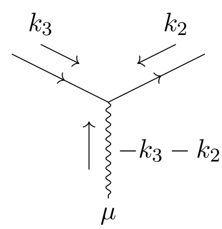


Figure 10.1: Feynman diagram associated with the Feynman rule, (10.4).

Recall that all the momenta are incoming.

If we take the time arrow to move in the left-to-right direction, the diagram above represents[1] the creation of a particle (a) with momentum k_3 that emits (or absorbs) a photon and continues propagating forward in time. The outgoing physical particle would have momentum $-k_2$. Note that the sign of k_2^0 gets reversed to become positive and, therefore, physical. This is not the only possibility, though. This very same vertex can represent different processes if we rotate it in different ways. If we take the time arrow to move in the right-to-left direction, the diagram above represents the creation of an antiparticle (b) which emits (or absorbs) a photon and continues forward in time. The same diagram can also represent the virtual creation of a particle-antiparticle pair, or the annihilation of a particle-antiparticle pair. In all these processes, charge is conserved but not the particle content. Still, they all originate from the very same vertex. This is a consequence of the implementation of relativity in QFTs.

We now consider $\delta\mathcal{L}^{(2)}$:

$$\delta\mathcal{L}^{(2)} = \int d^4x_1 \dots d^4x_4 A^{\mu_1}(x_1) A^{\mu_2}(x_2)\phi^*(x_3)\phi(x_4) \underbrace{\left(e^2 g_{\mu_1\mu_2} \prod_{i=1}^{4} \delta^{(4)}(x-x_i)\right)}_{\alpha_{\mu_1\mu_2}(x,x_1,\dots,x_4)} . \tag{10.5}$$

[1]Properly embedded in a full Feynman diagram so one does not get zero because of energy-momentum conservation for physical particles.

The Feynman rule for the vertex reads

$$I = i \times \alpha = 2ig_{\mu_1\mu_2}e^2 \,, \tag{10.6}$$

and the associated Feynman diagram reads

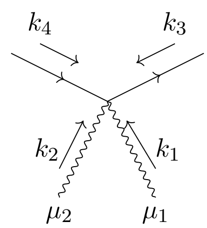


Figure 10.2: Feynman diagram associated with the Feynman rule, (10.6).

We do not write it explicitly but remember that the vertices have associated an energy-momentum conservation Dirac delta. In this case, the "complete" vertex would be

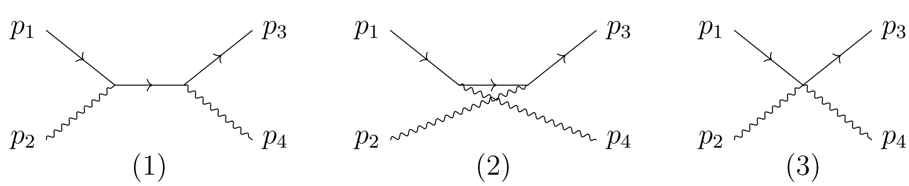
$$[2ig_{\mu_1\mu_2}e^2] \cdot (2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4} k_i\right) \,. \tag{10.7}$$

Since this Feynman rule is independent of the momenta, and symmetric under the interchange of μ_1 and μ_2, possible ambiguities due to misidentification of the momenta or indices vanish.

10.2 Compton Scattering with Pions to $O(e^2)$

We now consider the reaction $\pi^- + \gamma \to \pi^- + \gamma$. We take the π^- to be the particle and π^+ the antiparticle. The momenta are: p_1 (incoming π^-), p_2 (incoming γ), p_3 (outgoing π^-), p_4 (outgoing γ). r and r' are the helicities of the incoming and outgoing photons, respectively.

There are three Feynman diagrams that contribute to the S-matrix at $O(e^2)$:



The first two diagrams already appeared in the analogous computation for the $\lambda\phi^3$ theory discussed in Secs. 6.2 and 6.3. They correspond to Figs. 6.2 and 6.3. The topology Fig. 6.4 does not show up however. Diagram (3) above has the topology of a $\lambda\phi^4$ vertex. Note that, in the above diagrams, the momenta p_i represent the physical momentum of the particles. Therefore, p_1 and p_2 are incoming, while p_3 and p_4 are outgoing. This should be taken into account when using the Feynman rules to compute the S-matrix.

We now translate these diagrams into the quantity $i\mathcal{M}$. Recall that $i\mathcal{M}$ omits the factor $(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$. We obtain

$$i\mathcal{M} = (\text{external particles}) \times (\text{vertices}) \times (\text{propagator})\,. \tag{10.8}$$

Using the Feynman rules we computed in the previous section, we obtain

$$\begin{aligned} i\mathcal{M}_1 &= 1 \cdot \epsilon_\mu(p_2, r) \cdot 1 \cdot \epsilon^*_\nu(p_4, r') \\ &\quad \times [(-ie)(p_1 + p_1 + p_2)^\mu] \frac{i}{(p_1 + p_2)^2 - m_\pi^2 + i\eta} [(-ie)(p_3 + p_3 + p_4)^\nu] \\ &= -ie^2 \frac{[(2p_1 + p_2) \cdot \epsilon(p_2, r)][(2p_3 + p_4) \cdot \epsilon^*(p_4, r')]}{(p_1 + p_2)^2 - m_\pi^2 + i\eta}\,. \end{aligned} \tag{10.9}$$

Similarly,

$$i\mathcal{M}_2 = -ie^2 \frac{[(2p_1 - p_4) \cdot \epsilon^*(p_4, r')][(2p_3 - p_2) \cdot \epsilon(p_2, r)]}{(p_1 - p_4)^2 - m_\pi^2 + i\eta}\,. \tag{10.10}$$

Finally,

$$i\mathcal{M}_3 = 1 \cdot \epsilon^\mu(p_2, r) \cdot 1 \cdot \epsilon^\nu(p_4, r') \times [2ig_{\mu\nu}e^2] = 2ie^2\epsilon(p_2, r) \cdot \epsilon^*(p_4, r')\,. \tag{10.11}$$

So, the total S-matrix element reads

$$i\mathcal{M} = i\mathcal{M}_1 + i\mathcal{M}_2 + i\mathcal{M}_3 + O(e^4)\,. \tag{10.12}$$

This expression can be simplified using the fact that $\epsilon(k_i) \cdot k_i = 0$ for physical photons (since $\epsilon^0 = 0$, and $\epsilon \cdot \mathbf{k} = 0$):

$$i\mathcal{M} = ie^2\epsilon_2^\mu\epsilon_4^{*\nu}T_{\mu\nu} \quad \text{with} \quad T_{\mu\nu} = -\frac{4p_{1,\mu}p_{3,\nu}}{(p_1 + p_2)^2 - m_\pi^2} - \frac{4p_{3,\mu}p_{1,\nu}}{(p_1 - p_4)^2 - m_\pi^2} + 2g_{\mu\nu}\,. \tag{10.13}$$

Let us now check gauge invariance, i.e. the theorem stated in Sec. 9.2.3. We will show the invariance of the S-matrix under the transformation

$$\epsilon^\mu(k_i) \to \epsilon^\mu(k_i) + \alpha k_i^\mu\,, \tag{10.14}$$

where k_i^μ is on-shell ($k_0 = |\mathbf{k}|$). The first observation is that $\epsilon(k_i) \cdot k_i = 0$ holds true after this transformation, since $k_i^2 = 0$. Therefore, we can still use Eq. (10.13). After the transformation (10.14), we have

$$(\epsilon_2^\mu + \alpha\, p_2^\mu)\,(\epsilon_4^\nu + \beta\, p_4^\nu)\,T_{\mu\nu} = \epsilon_2^\mu\epsilon_4^\nu T_{\mu\nu} + \beta\,\epsilon_2^\mu T_{\mu\nu}p_4^\nu + \alpha\,\epsilon_4^\nu T_{\mu\nu}p_2^\mu + \alpha\beta\, p_2^\mu p_4^\nu T_{\mu\nu}\,.$$

Exercise

Show that

$$p_2^\mu T_{\mu\nu}\epsilon_4^{*\nu} = 0\,; \qquad p_4^\nu T_{\mu\nu}\epsilon_2^\mu = 0\,; \quad p_2^\mu p_4^\nu T_{\mu\nu} = 0\,. \tag{10.15}$$

Hint:

$$p_2^\mu T_{\mu\nu} = 2(p_4)_\nu\,. \tag{10.16}$$

This result shows that the S-matrix is invariant under the transformation (10.14). We stress that, to obtain this result, we need the contributions of the three diagrams. Gauge invariance does not hold on a diagram-by-diagram basis.

10.2.1 Compton Cross Section

The computation of the Compton-scattering cross section is usually done in the Lab frame, considering the pion at rest. In this reference frame, the expression for the matrix element computed in Eq. (10.13) simplifies considerably if computed in the Coulomb gauge since $p_1 \cdot \epsilon = 0$:

$$i\mathcal{M} = 2ie^2\epsilon(p_2, r)\cdot\epsilon^*(p_4, r') \qquad \text{(Lab frame and Coulomb gauge)}\,. \tag{10.17}$$

The phase space integral is similar to the one in Sec. 6.2.1 for $\lambda\phi^3$. The difference is that we are now interested in doing the computation in the Lab frame. Therefore, we want to emphasize the difference between the pion and the photon. To do so, we rename the momenta of each particle: $p_1 \to p$, $p_2 \to k$, $p_3 \to p'$ and $p^4 \to k'$.

We want to compute ($m_1 = m$ and $m_2 = 0$):

$$d\sigma = \frac{1}{2\lambda^{1/2}(s, m_1^2, m_2^2)}(2\pi)^4\delta^{(4)}(p_f - p_i)|\langle f|\mathcal{M}|i\rangle|^2 d\tilde{p}' d\tilde{k}'\,. \tag{10.18}$$

In our case, $2\lambda^{1/2}(s, m_1^2, m_2^2) = 4p\cdot k = 4mw_k$. The first equality holds in any reference frame, and the second in the Lab frame. We next want to determine k' as a function of k in the Lab frame. Since $k' - k = p - p'$, we have $(k'-k)^2 = (p-p')^2$ or

$$k'^2 + k^2 - 2k\cdot k' = p'^2 + p^2 - 2p'\cdot p\,,$$

which simplifies to

$$-2(w_k w_{k'} - w_k w_{k'}\cos\theta) = 2m^2 - 2m\omega_{p'} = 2m(m - \omega_{p'}) = 2m(w_{k'} - w_k) \tag{10.19}$$

in the Lab frame, and we obtain

$$w_{k'} = \frac{w_k}{1 + \frac{w_k}{m}(1-\cos\theta)}\,, \tag{10.20}$$

or, equivalently,

$$\frac{1}{w_{k'}} - \frac{1}{w_k} = \frac{1}{m}(1 - \cos\theta)\,. \tag{10.21}$$

Next, we perform the phase space integration of the differential cross section up to the solid angle of the outgoing photon. Working in the Lab frame, we obtain (recall that $w_k = |\mathbf{k}|$ and $w_{k'} = |\mathbf{k}'|$)

$$\begin{aligned}
&\int d^d\mathbf{p}' d^d\mathbf{k}' \delta^{(4)}(p' + k' - p - k) \qquad (10.22)\\
&= \int d\Omega_{k'} |\mathbf{k}'|^2 d|\mathbf{k}'| \delta(\sqrt{(\mathbf{k} - \mathbf{k}')^2 + m^2} + |\mathbf{k}'| - m - |\mathbf{k}|)\\
&= \int d\Omega \frac{|\mathbf{k}'|^2}{|1 + \frac{d\omega_{p'}}{dw_{k'}}|} = \int d\Omega \frac{w_{p'} w_{k'}^3}{m w_k}\,.
\end{aligned}$$

Using this result, we obtain

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{Lab} = \frac{\alpha^2}{m^2}\left(\frac{w_{k'}}{w_k}\right)^2 (\epsilon' \cdot \epsilon)^2\,,} \tag{10.23}$$

where $\epsilon' \cdot \epsilon$ is the polarization. This is the Klein-Nishima formula for scalar particles.

Let us now see how this expression simplifies depending on our knowledge of the polarizabilities of the initial/final photon.

1) No initial helicity. We first assume that we do not know the helicity of the initial photon. In such a scenario, the photon is described by a density matrix, and we must average over the helicities of the incoming photon. Therefore, what we measure is the following quantity:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\left.\frac{d\sigma}{d\Omega}\right|_{\epsilon_1} + \left.\frac{d\sigma}{d\Omega}\right|_{\epsilon_2}\right)\,, \tag{10.24}$$

and it will be sufficient to calculate $\frac{1}{2}[(\epsilon' \cdot \epsilon_1)^2 + (\epsilon' \cdot \epsilon_2)^2]$ to obtain it. To do so, we use the following trick: We decompose four-vectors in a convenient basis. It is clear that $(\frac{p}{m}, \epsilon_1, \epsilon_2, \hat{k})$ is an "orthonormal" basis of $\mathbb{R}^4$, where

$$\hat{k} \equiv \frac{1}{w_k}\begin{pmatrix} 0 \\ \mathbf{k} \end{pmatrix}\,, \quad \text{and} \quad \frac{p}{m} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}\,.$$

Then, for any four-vector u, we have:

$$u = (-u \cdot \epsilon_1)\epsilon_1 + (-u \cdot \epsilon_2)\epsilon_2 + (u \cdot \hat{k})\hat{k} - (u \cdot \frac{p}{m})\frac{p}{m}\,, \tag{10.25}$$

and

$$u^2 = -(u \cdot \epsilon_1)^2 - (u \cdot \epsilon_2)^2 + (u \cdot \hat{k})^2 + (u \cdot \frac{p}{m})^2 \,. \tag{10.26}$$

In particular, for $u = \epsilon'$:

$$\epsilon'^2 = -1 = -(\epsilon' \cdot \epsilon_1)^2 - (\epsilon' \cdot \epsilon_2)^2 - (\epsilon' \cdot \hat{k})^2 + 0 \,, \tag{10.27}$$

and

$$\frac{1}{2}[(\epsilon' \cdot \epsilon_1)^2 + (\epsilon' \cdot \epsilon_2)^2] = \frac{1}{2}(1 - (\epsilon' \cdot \hat{k})^2) \,. \tag{10.28}$$

So, then

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2} \left(\frac{k'}{k} \right)^2 (1 - (\epsilon' \cdot \hat{k})^2) \quad \text{(unpolarized initial photon)} \,. \tag{10.29}$$

2) Neither initial nor final helicity. Now, let us consider the case where we do not measure the polarization of the final photon either. This is realistic if we are only using a calorimeter as a detector. In this case:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\bigg|_{\epsilon'_1} + \frac{d\sigma}{d\Omega}\bigg|_{\epsilon'_2} \,. \tag{10.30}$$

Using the same identity (10.25) with $u = \hat{k}$ and the basis $(\frac{p}{m}, \epsilon'_1, \epsilon'_2, \hat{k}')$, we have

$$\hat{k}^2 = -1 = -(\hat{k} \cdot \epsilon'_1)^2 - (\hat{k} \cdot \epsilon'_2)^2 + (\hat{k} \cdot \hat{k}')^2 \,. \tag{10.31}$$

Then, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2} \left(\frac{w_{k'}}{w_k} \right)^2 (1 + \cos^2\theta) \quad \text{(unpolarized initial and final } \gamma) \,, \tag{10.32}$$

where θ is the angle between the incoming photon $\mathbf{k}$ and the outgoing $\mathbf{k}'$. In the NR limit, $|\mathbf{k}| \ll m$, $|\mathbf{k}'| \simeq |\mathbf{k}|$, and

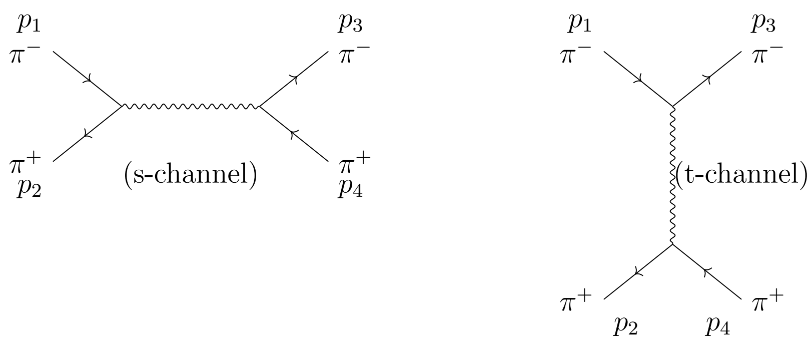
$$\frac{d\sigma}{d\Omega} \approx \frac{\alpha^2}{2m^2}(1 + \cos^2\theta) \,, \tag{10.33}$$

$$\sigma \approx \frac{8\pi\alpha^2}{3m^2} = \frac{8\pi r_e^2}{3} = \sigma_T \,, \tag{10.34}$$

where σ_T is the Thomson cross-section, and $r_e = \frac{\alpha}{m}$ is the classical radius of the electron. These NR formulae are universal and hold for massive particles of arbitrary spin, not only for scalars.

10.3 Pion-antipion Scattering

Let us now consider the reaction $\pi^+\pi^- \to \pi^+\pi^-$. We will calculate the matrix element to the first non-vanishing order $O(e^2)$. We have the following diagrams



The amplitude for the t-channel is

$$i\mathcal{M}^{(t)} = [(-ie)(p_1 + p_3)^\mu][(+ie)(p_2 + p_4)^\nu] \times D_{\mu\nu}(p_1 - p_3). \tag{10.35}$$

$D_{\mu\nu}$ is the photon propagator. We can take any gauge for it but one should remember that once it is fixed one has to use the same expression in all diagrams. For instance, we could take

$$D_{\mu\nu}(k) = \frac{(-i)}{(p_1 - p_3)^2}\left[g_{\mu\nu} - (1-\alpha)\frac{(p_1 - p_3)_\mu (p_1 - p_3)_\nu}{(p_1 - p_3)^2}\right] \tag{10.36}$$

or

$$D_{\mu\nu}(k) = \frac{i}{(p_1 - p_3)^2}\left[-g_{\mu\nu} - \frac{k^\mu k^\nu - (k\cdot n)\, n^\mu k^\nu - (k \cdot n)\, n^\nu k^\mu}{k^2}\right]. \tag{10.37}$$

Note that $(p_1 + p_3)\cdot(p_1 - p_3) = p_1^2 - p_3^2 = m_\pi^2 - m_\pi^2 = 0$, so the gauge-dependent part of the photon propagator is not relevant for $i\mathcal{M}^{(t)}$. A similar thing happens for $i\mathcal{M}^{(s)}$ in the s-channel diagram. Therefore, for this process, the gauge dependent part cancels on a diagram-by-diagram basis. Overall, both diagrams give the following results:

$$i\mathcal{M}^{(t)} = -ie^2\frac{(p_1 + p_3)\cdot(p_2 + p_4)}{(p_1 - p_3)^2 + i\eta}\,, \tag{10.38}$$

$$i\mathcal{M}^{(s)} = -ie^2\frac{(p_1 - p_2)\cdot(p_4 - p_3)}{(p_1 + p_2)^2}\,, \tag{10.39}$$

and the total amplitude is $i\mathcal{M} = i\mathcal{M}^{(t)} + i\mathcal{M}^{(s)}$.

Let us now obtain the NR limit. In the center-of-mass frame,

$$p^0_{\pi^-} \approx m_\pi + \frac{\mathbf{p}^2}{2m_\pi^2} \quad p^0_{\pi^+} \approx m_\pi + \frac{\mathbf{p}^2}{2m_\pi^2} \quad p'^0_{\pi^-} \approx m_\pi + \frac{\mathbf{p}'^2}{2m_\pi^2} \quad p'^0_{\pi^+} \approx m_\pi + \frac{\mathbf{p}'^2}{2m_\pi^2}.$$

In this limit, one can ignore the s-channel diagram and approximate the t-channel contribution to the following expression

$$i\mathcal{M} \approx i\mathcal{M}^{(t)} \approx i\frac{e^2(4m_\pi^2)}{(\mathbf{p}-\mathbf{p}')^2}\,. \tag{10.40}$$

This is indeed (proportional to) the Coulomb potential in momentum space. The factor $4m_\pi^2$ could be reabsorbed in the definition of the pion fields: $\phi(x) \to \sqrt{2m}\phi(x)$. We discuss this issue further in the next chapter.

With small variations of this exercise, we can consider many other possible processes. We list some of them in the following exercise.

Exercise

Compute the matrix element and the differential cross section at tree level of the following processes: $\pi^+\pi^+ \to \pi^+\pi^+$, $\pi^+K^- \to \pi^+K^-$ and $\pi^+\pi^- \to \gamma\gamma$. Consider their NR limit when possible.

It is interesting to see how, depending on the channel, some Feynman diagrams appear while others do not. It is also very important to recognize the universality of the Coulomb potential in the NR limit. It appears in all cases, only the charge of the scattered particles changes (so the interaction is repulsive for equal charged particles). As we will discuss further in the following chapter, this result also applies beyond scalar particles.

11 NRQED. Applications

[1] We now proceed to study NRQED, the theory introduced in Sec. 9.2. We begin by deriving the Feynman rules using the methods developed in Chapter 8. The equivalence between these Feynman rules and the direct application of Eq. (9.26) was already discussed in Secs. 9.2.2 and 9.2.3.

11.1 Feynman Rules of NRQED

Consider the Lagrangian (9.10) expanded:[2]

$$\mathcal{L} = \underbrace{\varphi^\dagger \left(i\partial_0 + \frac{\boldsymbol{\nabla}^2}{2m} \right) \varphi}_{\mathcal{L}_0} \underbrace{- e\varphi^\dagger A_0 \varphi}_{\mathcal{L}^{(A)}} \underbrace{- \frac{ie}{2m}\varphi^\dagger \boldsymbol{\nabla}\cdot\mathbf{A}\varphi) - \frac{ie}{2m}\varphi^\dagger \mathbf{A}\cdot\boldsymbol{\nabla}\varphi}_{\mathcal{L}^{(B)}} \underbrace{- \frac{e^2}{2m}\varphi^\dagger \mathbf{A}^2 \varphi}_{\mathcal{L}^{(C)}} . \tag{11.1}$$

We first calculate the propagator $D(k) = i\tilde{\mathcal{P}}^{-1}(k)$ following the prescriptions of Chapter 8. For this, we consider $\mathcal{L}_0$. We can identify $\mathcal{P}$ to be

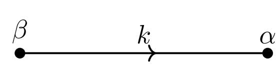
$$\mathcal{P}(x-y) = (i\partial_0 + \frac{\boldsymbol{\nabla}^2}{2m})\delta^{(4)}(x-y).$$

We need to invert it. Using

$$\int d^4y \mathcal{P}(x-y)\mathcal{P}^{-1}(y-z) = \delta^{(4)}(x-z),$$

we see that $(k_0 - c_k \frac{\mathbf{k}^2}{2m})\tilde{\mathcal{P}}^{-1}(k) = 1$. So, the propagator reads

$$D(k) = i\tilde{\mathcal{P}}^{-1}(k) = \frac{i}{k_0 - \frac{\mathbf{k}^2}{2m} + i\eta}\delta_{\alpha\beta} . \tag{11.2}$$



In the last equality we have added a Kronecker delta factor to indicate the spin of the particles, if any.

[1]For clarity, in this chapter, we omit introducing the $\hat{}$ notation for the operators of QFT but we will still keep them for operators in quantum mechanics.

[2]The photon propagator has already been worked out before. Therefore, we do not repeat the discussion here.

Now, let us calculate the different vertices. For $\mathcal{L}^{(A)}$, we have

$$\alpha^{(A)}(x; x_1, x_2, x_3) = -e\delta(x - x_1)\delta(x - x_2)\delta(x - x_3).$$

Doing the Fourier transform, we obtain $\tilde{\alpha}^{(A)} = -e$. So the Feynman rule reads

$$I^{(A)} = -ie\delta_{\alpha\beta} \tag{11.3}$$

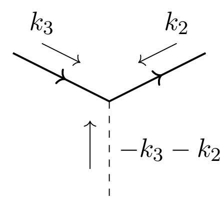


Figure 11.1: Feynman diagram associated with the Feynman rule, (11.3).

where, again, we have added a Kronecker delta factor to indicate that the interaction is diagonal in the spin space.

For $\mathcal{L}^{(B)}$, we first integrate by parts and rewrite it as

$$\mathcal{L}^{(B)} = -\frac{ie}{2m}(-(\boldsymbol{\nabla}\varphi^\dagger)\mathbf{A}\varphi + \varphi^\dagger\mathbf{A}(\boldsymbol{\nabla}\varphi)).$$

Exercise

Show that the Feynman rule of this vertex reads

$$I^{(B)} = \frac{ie}{2m}(\mathbf{k}_3 - \mathbf{k}_2)\delta_{\alpha\beta}\,. \tag{11.4}$$

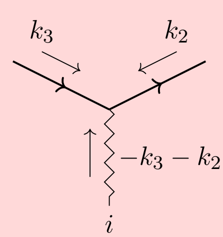


Figure 11.2: Feynman diagram associated with the Feynman rule, (11.4).

Finally, let us consider $\mathcal{L}^{(C)}$. Due to the combinatorics (there are two A's), and because $\tilde{\alpha} = -\frac{e^2}{2m}\delta^{ij}$, we have:

$$I^{(C)} = -\frac{ie^2}{m}\delta^{ij}\delta_{\alpha\beta}\,. \tag{11.5}$$

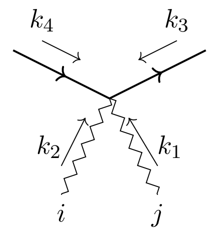
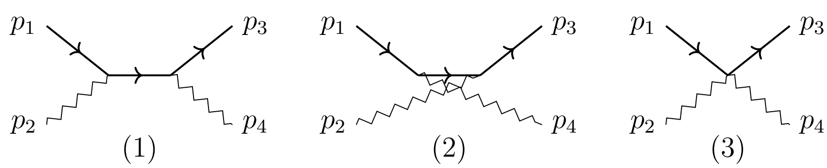


Figure 11.3: Feynman diagram associated with the Feynman rule, (11.5).

For all these Feynman rules, the direction of the NR particle is fixed. Therefore, we cannot rotate the diagram to generate new physical processes as we did in the relativistic case (see the discussion below Fig. 10.4). Note, however, that this qualification does not apply to the photon part of the Feynman rule, which can be deformed at will, as in the relativistic case.

11.2 Compton Scattering to $O(e^2)$ in NRQED

For the process $\pi^- + \gamma \to \pi^- + \gamma$ to order e^2, we have three diagrams:



Using the Feynman rules computed above and the methodology discussed in Chapter 8, we can construct the matrix element. Let us compare the importance of these terms. For each vertex, we have a factor $1/m$. So, at low energies, the diagram (3) gives the leading contribution to the S-matrix:

$$i\mathcal{M} = -i\frac{e^2}{m}\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_4 \quad \text{(low energy)}\,. \tag{11.6}$$

But in the relativistic case, we had $i\mathcal{M}^{(3)} = 2ie^2\epsilon_2 \cdot \epsilon_4 = -2ie^2\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_4$ (to ease computations, we use Coulomb gauge and work in the Lab frame). The factor of $\frac{1}{2m}$ difference is due to the different convention for the normalization of the creation and annihilation operators and does not affect physical results. We discuss this later.

In fact, this result is universal, independent of the spin of the massive particle. For a particle with spin, the result is obtained by multiplying Eq. (11.6) by the identity matrix in the spin space, i.e. a Kronecker delta $\delta_{\alpha\beta}$. This is one example of what is called a low-energy theorem.

11.3 Connection to the NR Quantum Theory

In this section, we discuss the relations between QED, NRQED, the NR constructions of QFTs developed in Chapter 2 and NR quantum mechanics courses in general. For the latter, we have in mind a course based on books like [12], for instance.

11.3.1 Connecting the Relativistic and NR Fields

We first connect scalar QED with NRQED. We have already seen that, up to an overall constant factor, we obtain the same result for the Compton scattering at low energies. We now quantify this similarity.

The conventions for the normalization of the commutation relations were chosen differently in the relativistic (R) and (NR) theories. They read as follows:

$$[a_R(\mathbf{k}), a_R^\dagger(\mathbf{k}')] = (2\pi)^d \delta^{(d)}(\mathbf{k}-\mathbf{k}') 2\omega_k \,, \tag{11.7}$$

$$[a_{NR}(\mathbf{k}), a_{NR}^\dagger(\mathbf{k}')] = (2\pi)^d \delta^{(d)}(\mathbf{k}-\mathbf{k}') \,. \tag{11.8}$$

We can relate them using the following definition: $a_{NR} \equiv \frac{a_R}{\sqrt{2w_k}} \simeq \frac{a_R}{\sqrt{2m}}$. We can then use the fields defined in Eq. (4.16) and the Lagrangian in Eq. (4.17). This Lagrangian, in the NR limit, can be approximated to

$$\mathcal{L} \simeq \phi^{(-)}_{\mathrm{NR}}(x) \left(i\partial_0 - m + \frac{\boldsymbol{\nabla}^2}{2m} \right) \phi^{(+)}_{\mathrm{NR}}(x) \,. \tag{11.9}$$

This very much resembles the Lagrangian in Eq. (2.1) with $V(\mathbf{x}) = 0$. There is a mismatch, though, due to the constant mass term in the Lagrangian. Therefore, we define (in the NR limit):

$$\varphi(x) = e^{imx^0} \phi^{(+)}_{\mathrm{NR}}(x) \simeq e^{imx^0} \sqrt{2m} \phi^{(+)}(x) \,. \tag{11.10}$$

This shifts the energy levels by a constant and finally matches the expression we had in Eq. (2.1) with $V(\mathbf{x}) = 0$. Using Eqs. (11.7), (11.8) and (11.10), we obtain the following relation between the S-matrix elements in both theories:

$$\frac{1}{2m} \langle 0| a_R \cdots (\cdots \phi \cdots) \cdots a_R^\dagger |0\rangle \simeq \langle 0| a_{NR} \cdots (\cdots \varphi/(\sqrt{2m}) \cdots) \cdots a_{NR}^\dagger |0\rangle \,. \tag{11.11}$$

This explains the factor $2m$ difference between Eq. (10.17) and Eq. (11.6).

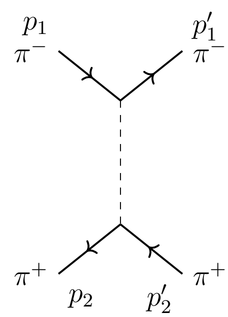
11.3.2 Electron-proton Scattering in NRQED

Let us now discuss the electron-proton scattering at low energies. The result is the same as the $\pi^-\pi^+ \to \pi^-\pi^+$ (or $\pi^- K^+ \to \pi^- K^+$) scattering at low energies multiplied by the identity in spin of the electron and proton: $\delta_{\sigma_1 \sigma_1'} \delta_{\alpha_2 \alpha_2'}$. Such process can be determined using the following Lagrangian density that describes a NR charged pion (φ) and antipion/antikaon (χ_c) interacting with photons

$$\begin{aligned} \mathcal{L} &= \varphi^\dagger(t,\mathbf{x})\left\{iD_0 + \frac{\mathbf{D}^2}{2m_1}\right\}\varphi(t,\mathbf{x}) \\ &+\chi_c^\dagger(t,\mathbf{x})\left\{iD_0^c + \frac{\mathbf{D}_c^2}{2m_2}\right\}\chi_c(t,\mathbf{x}) \\ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \end{aligned} \tag{11.12}$$

where $iD_0^c = i\partial_0 + eA_0$ and $i\mathbf{D}^c = i\boldsymbol{\nabla} - e\mathbf{A}$.

The leading contribution in the $1/m$ expansion comes from the following diagram



At leading order in the $1/m$ expansion, we obtain ($\mathbf{k} = \mathbf{p}_1 - \mathbf{p}_1'$)

$$i\mathcal{M} = ie^2\frac{1}{\mathbf{k^2}}\,. \tag{11.13}$$

This is exact in the Coulomb gauge. Nevertheless, we emphasize that this result holds true regardless of the gauge used for the longitudinal component of the photon field. In a general gauge, the denominator of the propagator has a k^0 dependence: $k_0^2 - \mathbf{k}^2$. However, this dependence can be neglected in the NR limit since $k_0^2 \ll \mathbf{k}^2$.

If we compare Eq. (11.13) and Eq. (10.40), we observe that they are equal up to a factor of $(2m)^2$ (or $(2m_1)(2m_2)$ for different particles). This is consistent with adding a factor $\sqrt{2m}$ for each external particle, as discussed in Sec. 11.3.1.

As we have already seen, the S-matrix is proportional to the Coulomb potential in momentum space. We would now like to connect this observation with a NR quantum mechanical formulation. We first observe that this contribution to the S-matrix can be obtained from the original H_I using only Eq. (9.28):

$$\delta H_I = \frac{e^2}{8\pi}\int d^d\mathbf{x}\, d^d\mathbf{y}\, \rho(\mathbf{x})\,\rho(\mathbf{y})\,\frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{11.14}$$

For this to hold, we must generalize the discussion in Sec. 9.2 to the case where several NR particles with (possibly) different charges are present. This means generalizing the charge density in Eq. (9.16) to the following expression:

$$\rho = Q_\varphi \varphi^\dagger \varphi + Q_{\chi_c} \chi_c^\dagger \chi_c + \cdots \,, \tag{11.15}$$

where the dots represent additional NR particles that may be present in the system. The overall normalization of the charge, e, has been factored out in the interaction term. Therefore, in nature, we typically have that Q_X takes integer or fractional values. In our case, we take $Q_\varphi = 1$ and $Q_{\chi_c} = -1$.

Eq. (11.14) is a non-local (in space) four field operator. This very much resembles the interaction term we had in Eq. (2.77):

$$H_I = \int d^d\mathbf{x}_1 d^d\mathbf{x}_2 (\varphi^\dagger \chi_c^\dagger)(\mathbf{x}_1) V(|\mathbf{x}_1 - \mathbf{x}_2|)(\chi_c \varphi)(\mathbf{x}_2)\,, \tag{11.16}$$

where ($\alpha = e^2/(4\pi)$)

$$V = -\frac{\alpha}{|\mathbf{x}|} \quad \text{and} \quad \tilde{V} = -\frac{e^2}{|\mathbf{k}|^2}\,. \tag{11.17}$$

Strictly speaking, Eq. (11.14) is more general and also applies to different channels.[3] The equivalence with Eq. (11.16) holds only when we consider the particular channel where we have one particle φ and one antiparticle χ_c. We discuss this further below.

Let us keep the potential general for a moment. If we compute the S-matrix element to order V using Eq. (11.16), we obtain

$$\langle f|(-i)\int_{-\infty}^{\infty} d\tau H_I(\tau)|i\rangle \approx (2\pi)^4 \delta^{(4)}(p_f - p_i)(-i\tilde{V}(\mathbf{k}))\,, \tag{11.19}$$

where $\mathbf{k} = \mathbf{p}_1 - \mathbf{p}_1'$ in the center-of-mass frame. Note that we have translational invariance for the center-of-mass coordinate. Here, $\tilde{V}$ is the Fourier transform of $V(|\mathbf{x}_1 - \mathbf{x}_2|)$. Therefore, at tree level, we can make the following assignment in the NR limit:

$$\boxed{\tilde{V} = i(i\mathcal{M})\,.} \tag{11.20}$$

[3] Eq. (11.14) also generates extra terms when the four fields represent the same particle. In this case, Eq. (11.14) can be split into the normal-ordered term, which would correspond to Eq. (11.16) for the particle-particle channel, plus one extra contribution that has the following form:

$$\sim V(\mathbf{0})\varphi^\dagger \varphi\,. \tag{11.18}$$

This term is formally infinity. Nevertheless, it can be reabsorbed into the definition of m, which goes into the e^{imx^0} phase that appeared in Eq. (11.10). This provides a first, primitive, example of renormalization.

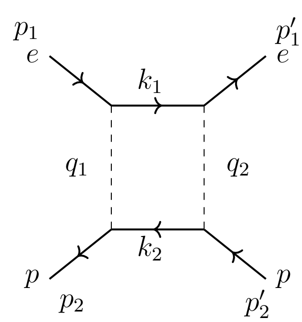
We have already seen this in QED with the Coulomb potential. Nevertheless, one could also consider replacing the photon with a massive boson. Let us call it Z, with a coupling e. The associated potential would then have a Yukawa form:

$$\tilde{V} = -\frac{e^2}{\mathbf{k}^2 + m^2} \quad \text{or} \quad V = -\frac{\alpha'}{|\mathbf{x}|} e^{-m|\mathbf{x}|}. \tag{11.21}$$

Eq. (11.20) is nothing but the **Born approximation** that appears in quantum mechanics courses. But we can go beyond this. We saw in Exercise 1 in Sec. 2.3 that, in a quantum mechanical formulation, the Lagrangian (2.77) describes a physical system composed of two NR particles interacting via a potential $V(\mathbf{x})$. The solution of the associated Schrödinger equation, Eq. (2.79), incorporates all-order terms in the $V(\mathbf{x})$ expansion, whereas the perturbative computation of the S-matrix we performed in Eq. (11.20) is only valid to $\mathcal{O}(V)$. We now look at how the perturbative computation of the S-matrix to all orders in V recovers the Schrödinger equation. We discuss this issue in the next section.

11.3.3 Lippmann-Schwinger Equation

We now want to go beyond $\mathcal{O}(e^2)$. At order $O(e^4)$, we have several possible diagrams. In the NR limit, the leading one is the following



Let us now compute it. We will also use it as an example of a loop computation. We will evaluate this matrix element for the general case of two NR particles with different masses (for instance, an electron and a proton). We neglect the spin of the particles, as the result is diagonal in spin space.

We will do the computation in the Coulomb gauge. In this gauge, the longitudinal photon propagator is equivalent to the interaction term

$$\delta H_I = \frac{e^2}{8\pi} \int d^d\mathbf{x} d^d\mathbf{y} \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}.$$

The contribution to the S-matrix from this diagram then reads

$$\begin{aligned}\langle f|S|i\rangle &\doteq (-ie)^2(ie)^2\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4} \\ &\times(2\pi)^4\delta^{(4)}(p_1-q_1-k_1)(2\pi)^4\delta^{(4)}(p_2+q_1-k_2) \\ &\times(2\pi)^4\delta^{(4)}(k_1-p_1'-q_2)(2\pi)^4\delta^{(4)}(k_2+q_2-p_2') \\ &\times\frac{i}{k_1^0-\frac{\mathbf{k}_1^2}{2m_1}+i\eta}\frac{i}{k_2^0-\frac{\mathbf{k}_2^2}{2m_2}+i\eta}\frac{i}{\mathbf{q}_1^2}\frac{i}{\mathbf{q}_2^2}\,.\end{aligned}\tag{11.22}$$

The momenta p_1 and p_2 are incoming and p_1' and p_2' are outgoing. The direction of q_1 and q_2 is irrelevant. The k_1 and k_2 flow in the left-to-right direction.

We have four Dirac deltas, one for each vertex. We also have four integrals, one for each propagator. We will be able to remove three integrals using the Dirac deltas but one Dirac delta will remain, producing the overall energy-momentum conservation. This implies that one of the integrals will remain. This is the loop. Overall, we get

$$\begin{aligned}&\langle f|S|i\rangle\doteq(2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2') \\ &\times(-e^2)^2\int\frac{d^4q}{(2\pi)^4}\frac{i}{\mathbf{q}^2}\frac{i}{(\mathbf{p}_1-\mathbf{p}_1'-\mathbf{q})^2} \\ &\times\frac{i}{(p_1^0-q^0)-\frac{(\mathbf{p}_1-\mathbf{q})^2}{2m_1}+i\eta}\frac{i}{(p_2^0+q^0)-\frac{(\mathbf{p}_2+\mathbf{q})^2}{2m_2}+i\eta}\,.\end{aligned}\tag{11.23}$$

The integration over q^0 can easily be done using Cauchy's theorem:

$$\begin{aligned}&\int\frac{dq^0}{(2\pi)}\frac{i}{(p_1^0-q^0)-\frac{(\mathbf{p}_1-\mathbf{q})^2}{2m_1}+i\eta}\frac{i}{(p_2^0+q^0)-\frac{(\mathbf{p}_2+\mathbf{q})^2}{2m_2}+i\eta} \\ &=\frac{i}{(p_1^0+p_2^0)-\frac{(\mathbf{p}_1-\mathbf{q})^2}{2m_1}-\frac{(\mathbf{p}_2+\mathbf{q})^2}{2m_2}+i\eta}\,.\end{aligned}\tag{11.24}$$

Using this result, and rewriting the integral in terms of the center-of-mass coordinates in momentum space ($\eta_1=m_1/(m_1+m_2)$ and $\eta_2=m_2/(m_1+m_2)$):

$$\mathbf{p}_1=\eta_1\mathbf{P}+\mathbf{p}\qquad\mathbf{p}_2=\eta_2\mathbf{P}-\mathbf{p}\,,$$

we obtain

$$\begin{aligned}&\langle f|S|i\rangle\doteq(2\pi)^4\delta(E-E')\delta^{(3)}(\mathbf{P}-\mathbf{P}')e^4 \\ &\times\int\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{(\mathbf{p}-\mathbf{q})^2}\frac{1}{(\mathbf{p}'-\mathbf{q})^2}\frac{i}{E-\frac{\mathbf{P}^2}{2M}-\frac{\mathbf{q}^2}{2m_r}+i\eta}\,,\end{aligned}\tag{11.25}$$

where $E = p_1^0 + p_2^0$, $M = m_1 + m_2$ is the total mass, and $m_r = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass.

We are now in a position to connect with the notation used in NR quantum mechanics.

Reminder of notation in quantum mechanics

In position space, we have the following equalities

$$\langle \mathbf{x}'|\mathbf{x}\rangle = \delta^{(d)}(\mathbf{x}-\mathbf{x}')\,, \qquad \langle \mathbf{x}'|\hat{V}(\hat{\mathbf{x}})|\mathbf{x}\rangle = \delta^{(d)}(\mathbf{x}-\mathbf{x}')V(\mathbf{x})\,, \tag{11.26}$$

whereas in momentum space we have

$$\langle \mathbf{p}'|\mathbf{p}\rangle = (2\pi)^d\delta^{(d)}(\mathbf{p}-\mathbf{p}')\,, \qquad \langle \mathbf{p}'|\hat{V}(\hat{\mathbf{x}})|\mathbf{p}\rangle = \tilde{V}(\mathbf{p}'-\mathbf{p})\,. \tag{11.27}$$

This means that the free propagator is local in momentum space (something that we already knew):

$$\begin{aligned}&\langle \mathbf{P}'|\langle \mathbf{p}'|\frac{i}{E-\frac{\hat{\mathbf{P}}^2}{2M}-\frac{\hat{\mathbf{p}}^2}{2m_r}+i\eta}|\mathbf{P}\rangle|\mathbf{p}\rangle \qquad (11.28)\\ &= (2\pi)^d\delta^{(d)}(\mathbf{P}-\mathbf{P}')(2\pi)^d\delta^{(d)}(\mathbf{p}-\mathbf{p}')\frac{i}{E-\frac{\mathbf{P}^2}{2M}-\frac{\mathbf{p}^2}{2m_r}+i\eta}\,,\end{aligned}$$

where $|\mathbf{P}\rangle|\mathbf{p}\rangle = a^\dagger_{NR}(\mathbf{p}_1)a^\dagger_{NR}(\mathbf{p}_2)|0\rangle$.

Using this notation, we can combine Eqs. (11.13) and (11.25) in the following way:

$$\begin{aligned}&\langle \mathbf{P}'|\langle \mathbf{p}'|\hat{S}|\mathbf{P}\rangle|\mathbf{p}\rangle = -i(2\pi)\delta(E-E') \qquad (11.29)\\ &\times\langle \mathbf{P}'|\langle \mathbf{p}'|\left(\hat{V}+\hat{V}\frac{1}{E-\hat{T}+i\eta}\hat{V}+\mathcal{O}(e^6)\right)|\mathbf{P}\rangle|\mathbf{p}\rangle\,,\end{aligned}$$

where we have defined:

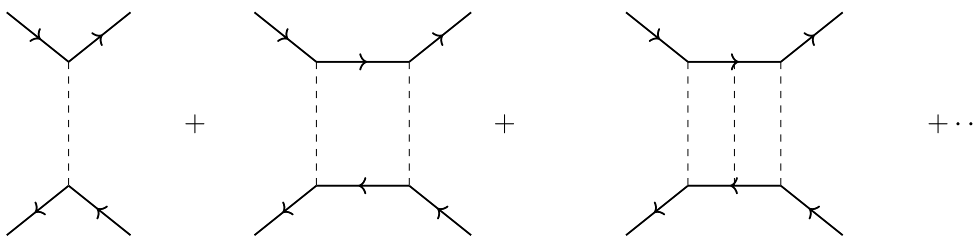
$$\hat{T} \equiv \hat{T}_{\mathbf{X}} + \hat{T}_{\mathbf{x}} = \frac{\hat{\mathbf{P}}^2}{2M} + \frac{\hat{\mathbf{p}}^2}{2m_r}\,, \tag{11.30}$$

which is the free kinetic energy. We also have that $V = -\frac{\alpha}{|\mathbf{x}|}$. Nevertheless, the discussion can be made for a general potential.

The result obtained in Eq. (11.29) can be extrapolated to an all-order result in the NR limit:

$$\langle f|\hat{S}|i\rangle = -i(2\pi)\delta(E-E') \tag{11.31}$$
$$\times\langle \mathbf{P}'|\langle \mathbf{p}'|\left(\hat{V}+\hat{V}\frac{1}{E-\hat{T}+i\eta}\hat{V}+\hat{V}\frac{1}{E-\hat{T}+i\eta}\hat{V}\frac{1}{E-\hat{T}+i\eta}\hat{V}+\cdots\right)|\mathbf{P}\rangle|\mathbf{p}\rangle$$
$$\boxed{= -i(2\pi)\delta(E-E')\ \langle \mathbf{P}'|\langle \mathbf{p}'|\left(\hat{V}+\hat{V}\frac{1}{E-\hat{T}-\hat{V}+i\eta}\hat{V}\right)|\mathbf{P}\rangle|\mathbf{p}\rangle\,.}$$

This formula is the **Lippmann-Schwinger** equation, which is the S-matrix realization of the Schrödinger equation. This is equivalent to the summation of the following infinite set of diagrams in the NR limit



Since the dependence on the total three-momentum of the system is trivial ($\mathbf{P}$ remains constant), one could work in the center-of-mass frame, where $\mathbf{P}=\mathbf{0}$, and omit its explicit dependence. Either way, it is interesting to note that the following propagator appears in the S-matrix element ($\hat{h}_{\mathbf{x}}=\hat{T}_{\mathbf{x}}+\hat{V}(\mathbf{x})$):

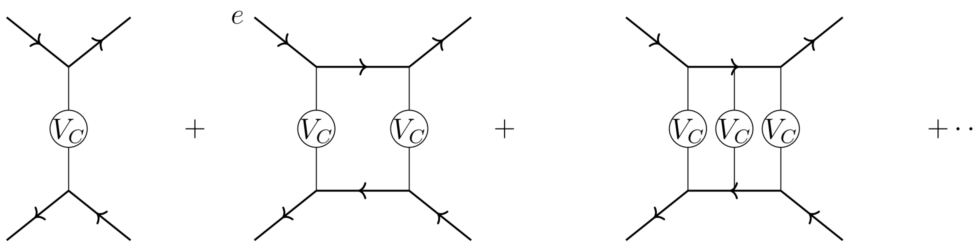
$$\langle \mathbf{x}'|\frac{1}{E-\hat{h}_{\mathbf{x}}+i\eta}|\mathbf{x}\rangle\,, \tag{11.32}$$

which satisfies the Green's function condition:

$$(\hat{h}_{\mathbf{x}}-E)\langle \mathbf{x}'|\frac{1}{E-\hat{h}_{\mathbf{x}}+i\eta}|\mathbf{x}\rangle = -\delta^{(3)}(\mathbf{x}'-\mathbf{x})\,. \tag{11.33}$$

This means that, by studying the analytic properties of the S-matrix, one can find the poles of the propagator, which is tantamount to obtaining the energy levels of hydrogen (or whatever analogous system we are studying). Alternatively, solving the Schrödinger equation provides a representation of the hydrogen Green's function, which can then be used in the Lippmann-Schwinger equation and in the S-matrix.

On the diagrammatic side, it is better to represent the Lippmann-Schwinger equation directly in terms of the potential (the Coulomb potential in our case):

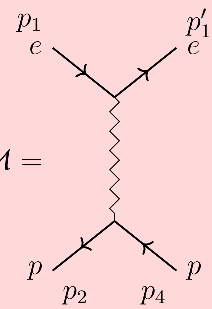


Once drawn this way, the result is exact and gauge independent (unlike when drawn in terms of the longitudinal component of the gauge boson). This formulation is also convenient for incorporating corrections to the Schrödinger equation, as we will discuss next. The key point is that a fixed-order computation (a tree level one in our case) generates an all-order result by using the Lippmann-Schwinger equation. Let us illustrate this with the following exercise:

Exercise

Compute the following diagram for the vertices that appear in Eq. (11.12) in the NR limit

$i\delta\mathcal{M} =$



The zigzag propagator indicates a transverse photon.

One can generalize Eq. (11.20) to include this relativistic correction using

$$i\delta\mathcal{M} = -i\delta V.$$

This correction will also generate an infinite number of diagrams. These can be concisely incorporated in the Lippmann-Schwinger equation by simply modifying the potential:

$$\boxed{V_C \to V_C + \delta V\,.} \tag{11.34}$$

Interpreted in this way, one can obtain the effect associated with $i\delta\mathcal{M}$ using standard methods from quantum mechanics courses.[4]

[4]A warning though. When corrections to the Coulomb potential are considered, individual

11.4 NR Particles in an External Potential

In the previous section, we saw how to derive the Schrödinger equation for the hydrogen atom (and similar systems) from a QFT. We also discussed how corrections to the Coulomb potential are obtained. However, the connection with the formulation in Chapter 2 is not yet complete. In that chapter, we had a QFT made of free NR particles in an external potential. One can make the connection by considering the proton (or nucleus) to be infinitely heavy, so that it behaves as a fixed external source of electromagnetic interactions. We then have NR electrons, which can interact either among themselves or with physical photons. If we switch off these interactions, we end up with a system of free NR electrons in an external field. By keeping them, we will be able to deduce the interactions of these NR particles in an external Coulomb potential with photons (and therefore, indirectly, among themselves).

Let us quantify this discussion. Recall that we work in the interaction picture. We first consider the propagator for the heavy particle field χ_c (a proton, for instance):

$$\begin{aligned}\langle 0_{\chi_c}|T\{\chi_c(x')\chi_c^\dagger(x)\}|0_{\chi_c}\rangle &= \theta(x'^0-x^0)\langle 0_{\chi_c}|\chi_c(x')\chi_c^\dagger(x)|0_{\chi_c}\rangle \\ &= \int \frac{d^4k}{(2\pi)^4}e^{-ik(x'-x)}\frac{i}{k^0-\frac{\mathbf{k}^2}{2m_p}+i\eta}\mathbb{1}_{\text{no-}\chi_c}\,. \end{aligned} \tag{11.35}$$

In the limit $m_p \to \infty$, the propagator in position space becomes:

$$\begin{aligned}\langle 0_{\chi_c}|T\{\chi_c(x')\chi_c^\dagger(x)\}|0_{\chi_c}\rangle \xrightarrow{m_p\to\infty} &\int \frac{d^4k}{(2\pi)^4}e^{-ik(x'-x)}\frac{i}{k^0+i\eta} \\ &= \theta(x'^0-x^0)\delta^{(3)}(\mathbf{x}'-\mathbf{x})\mathbb{1}_{\text{no-}\chi_c}\,. \end{aligned} \tag{11.36}$$

This represents a particle that does not move, it remains static.

In the above equations, we are using the vacuum associated exclusively with the particle χ_c. This differs from the standard computation of a propagator, where the time-ordered operators are sandwiched between the full vacuum. Therefore, the expressions in Eqs. (11.35) and (11.36) are to be understood as identity operators acting on the Hilbert space of all particles *except* χ_c, which we have indicated with the operator $\mathbb{1}_{\text{no-}\chi_c}$.

Now, let us compute (generically, we will consider that $x'^0 > x^0$)

$$\langle 0_{\chi_c}|\chi_c(x')H\chi_c^\dagger(x)|0_{\chi_c}\rangle\,,$$

tree-level diagrams may be gauge dependent, and loop diagrams could also generate corrections to the potentials. A complete set of diagrams is therefore needed to obtain a meaningful correction to the potential.

where

$$H = H^{(0)}_{\chi_c} + H^{(0)}_{other} + H_I.$$

Since $m_p \to \infty$, $\hat{H}^{(0)}_{\chi_c} = 0$. $\hat{H}^{(0)}_{other}$ stands for the free Hamiltonian of the other degrees of freedom. It commutes with χ_c. Then:

$$\begin{aligned}\langle 0_{\chi_c}|\chi_c(x')H\chi_c^\dagger(x)|0_{\chi_c}\rangle &= \theta(x'^0 - x^0)\delta^{(3)}(\mathbf{x}' - \mathbf{x})H^{(0)}_{other}\mathbb{1}_{\text{no-}\chi_c} \\ &\quad + \langle 0_{\chi_c}|\chi_c(x')H_I{\chi_c}^\dagger(x)|0_{\chi_c}\rangle\,. \end{aligned} \tag{11.37}$$

Let us now focus on H_I. It can be split into the term that only depends on other degrees of freedom, H_I^{other}, and the term that mixes those degrees of freedom with χ_c. This latter term comes from Eq. (11.14), with the qualification that we set $Q_{\chi_c} = -Z$, allowing for a general heavy nucleus. We then obtain

$$\begin{aligned}&\langle 0_{\chi_c}|\chi_c(x')H_I{\chi_c}^\dagger(x)|0_{\chi_c}\rangle = \mathbb{1}_{\text{no-}\chi_c}\theta(x'^0 - x^0)\delta^{(3)}(\mathbf{x}' - \mathbf{x}) \\ &\times\left(H_I^{other} + \int d^d\mathbf{y}\left(\frac{-Ze^2}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\varphi^\dagger(\mathbf{y})\varphi(\mathbf{y})\right)\,.\end{aligned} \tag{11.38}$$

In this expression, the nucleus field has disappeared from the equation, effectively creating a Coulomb potential field of strength $Z\alpha$ located at $\mathbf{x}$. At the Lagrangian level this term can be incorporated using the following term:

$$-\,\delta L_I = \int d^d\mathbf{y}\varphi^\dagger(\mathbf{y})\frac{Z\alpha}{|\mathbf{y}-\mathbf{x}|}\varphi(\mathbf{y})\,, \tag{11.39}$$

and we can recover the Lagrangian we had in Eq. (2.1) of Chapter 2:

$$L^{(0)}_\varphi = \int d^d\mathbf{x}\varphi^\dagger(x^0,\mathbf{x})\left(i\partial_0 + \frac{\boldsymbol{\nabla}^2}{2m} - \frac{Z\alpha}{|\mathbf{x}|}\right)\varphi(x^0,\mathbf{x})\,, \tag{11.40}$$

where we have located the heavy nucleus at the origin.[5] On top of that, we can trivially include the interactions of the electron(s) with photons due to $\hat{H}_I^{other}$ by including the interaction terms, L_I, one had in Eq. (9.10). We illustrate this in the following sections.

Overall, the above derivation can be interpreted as creating an effective field theory of NR particles in an external field from the original S-matrix projected onto the one-proton sector:

$$S_{eff}\theta(x'^0 - x^0)\delta^{(3)}(\mathbf{x}' - \mathbf{x}) = \langle 0_{\chi_c}|T\{\chi_c(x')e^{-i\int dy^0 H_I^I(y^0)}{\chi_c}^\dagger(x)\}|0_{\chi_c}\rangle\,. \tag{11.42}$$

[5]We could easily generalize to the situation where we have several static charged particles located at positions $\mathbf{x}_i$ by changing the Coulomb potential to

$$-\frac{Z\alpha}{|\mathbf{x}|} \to -\sum_i \frac{Z_i\alpha}{|\mathbf{x}-\mathbf{x}_i|}\,. \tag{11.41}$$

11.4.1 Radiative Transitions of Hydrogen

The results obtained in the previous section allow us to compute the radiative transitions of hydrogen-like systems. Our starting point is the Lagrangian

$$L = \int d^d\mathbf{x}\varphi^\dagger(t,\mathbf{x})\left(i\partial_0 + \frac{\mathbf{D}^2}{2m} + \frac{Z\alpha}{|\mathbf{x}|}\right)\varphi(t,\mathbf{x}) - \int d^d\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{11.43}$$

where $\varphi(x)$ describes the hydrogen(-like) atom field and $i\mathbf{D} = i\boldsymbol{\nabla} + e\mathbf{A}$ is the covariant derivative, which we split in the following way:

$$L = L_\varphi^{(0)} + L_\gamma + L_I\,, \tag{11.44}$$

where $L_\varphi^{(0)}$ is the Lagrangian of free NR particles in an external potential (the Coulomb potential in our case) obtained in the previous section, i.e. Eq. (11.40). $L_\gamma = -\frac{1}{4}\int d^d\mathbf{x}F^2$, and the interaction term can be approximated by the following expression for the purposes of this section:

$$L_I \approx \int (\frac{-ie}{m})d^d\mathbf{x}\varphi^\dagger\mathbf{A}\cdot\boldsymbol{\nabla}\varphi \quad \text{(Coulomb gauge)}.$$

We will calculate the decay of hydrogen in the quantum state n into hydrogen in the quantum state n' plus a photon. Our initial state is

$$|i\rangle = |n\rangle|0_\gamma\rangle = a_n^\dagger|0_\varphi\rangle|0_\gamma\rangle = a_n^\dagger|0\rangle\,,$$

where $a_n^\dagger$ is the creation operator of a hydrogen-like state with quantum numbers n. We neglect recoil effects ($m_p \to \infty$). Therefore, we set its center-of-mass momentum to zero (both for the initial and final states).

Our final state is

$$|f\rangle = |n'\rangle \otimes a^\dagger(\mathbf{k},\sigma)|0_\gamma\rangle = a_{n'}^\dagger a_\sigma^\dagger(\mathbf{k})|0\rangle.$$

We now compute the S-matrix to $\mathcal{O}(e)$.

$$\begin{aligned}&\langle f|i\int dx^0 L_I^I(x^0)|i\rangle \qquad (11.45)\\ &= \int dx^0\left(\frac{-ie}{m}\right)\int d^d\mathbf{x}\langle n'|\langle 0_\gamma|a_\sigma(\mathbf{k})\varphi^\dagger(t,\mathbf{x})\mathbf{A}(t,\mathbf{x})\cdot(\boldsymbol{\nabla}_x\varphi(t,\mathbf{x}))|n\rangle|0_\gamma\rangle\,,\end{aligned}$$

where, as usual,

$$\mathbf{A}(x^0,\mathbf{x}) = \int d\tilde{q}\sum_{\sigma'}(\boldsymbol{\epsilon}_{\sigma'}a_{\sigma'}(\mathbf{q})e^{-iqx} + \boldsymbol{\epsilon}^*_{\sigma'}a^\dagger_{\sigma'}(\mathbf{q})e^{iqx})\,, \tag{11.46}$$

and

$$\varphi(x^0, \mathbf{x}) = \sum_n a_n e^{-iE_n x^0} \phi_n(\mathbf{x}) \, . \tag{11.47}$$

We can factor out the computation associated with the photon from the one associated with the bound state ($q^0 = E_q = w_q$):

$$\langle f|S|i\rangle = \left(\frac{-ie}{m}\right) \int dx^0 d^d\mathbf{x} \int d\tilde{q} \left(\sum_{\sigma'} \langle 0_\gamma | a_\sigma(\mathbf{k}) a^\dagger_{\sigma'}(\mathbf{q}) | 0_\gamma \rangle \boldsymbol{\epsilon}^*_{\sigma'}(\mathbf{q}) \right) e^{iqx}$$
$$\times \langle n'|\phi^\dagger(t,\mathbf{x})(\boldsymbol{\nabla}\phi(t,\mathbf{x}))|n\rangle = \dots$$
$$= \left(\frac{-ie}{m}\right)(2\pi)\delta(E_n - \omega_k - E_{n'})\boldsymbol{\epsilon}^*_\sigma(\mathbf{k}) \cdot \int d^d\mathbf{x} \phi^*_{n'}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} (\boldsymbol{\nabla}\phi_n(\mathbf{x})) \, . \tag{11.48}$$

This last integration can be written in regular quantum mechanics as

$$\int d^d\mathbf{x} \phi^*_{n'}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} (\boldsymbol{\nabla}\phi_n(\mathbf{x})) = i\langle n'|e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}}\hat{\mathbf{p}}|n\rangle.$$

This expression can be simplified using the multipole expansion: $e^{-i\mathbf{k}\cdot\mathbf{x}} = 1 + O(\alpha)$. This follows from the fact that $|\mathbf{k}| = w_k \sim m\alpha^2$ for the photon energy, whereas $1/|\mathbf{x}| \sim m\alpha$, for the inverse radius of the atom.

The S-matrix element can be written compactly as follows

$$\langle f|S|i\rangle = \langle f|i\rangle + i(2\pi)\delta(E_f - E_i)\langle f|\mathcal{M}|i\rangle \, , \tag{11.49}$$

where

$$\langle f|\mathcal{M}|i\rangle = -\frac{e}{m}\boldsymbol{\epsilon}^*_\sigma(\mathbf{k}) i\langle n'|\hat{\mathbf{p}}|n\rangle \, . \tag{11.50}$$

This result does not have the three-momentum conservation Dirac delta because the original Lagrangian (11.44) does not have translational invariance. Obviously, physics is translation invariant. The reason we do not see an explicit three-momentum conservation delta is because we have factored it out of the S-matrix element computed above. See Eq. (11.31) for an illustration of a place where we still explicitly write the total three-momentum Dirac delta in the S-matrix element.

Since there is no explicit translation symmetry, and the normalization of the states is different, we have to repeat the derivation for the cross sections and decays we did in Sec. 5.5. Here we perform a slightly different and simplified derivation. We now have

$$W_{f\leftarrow i} = (2\pi)\delta(0)(2\pi)\delta(E_f - E_i)|\langle f|\mathcal{M}|i\rangle|^2 \, , \tag{11.51}$$

where $E_f = E_{n'} + w_k$ and $E_i = E_n$. $\delta(0)$ is infinity so we have to look at its original expression to regularize it.

$$(2\pi)\delta(E_f - E_i) = \lim_{T\to\infty} \int_{-T/2}^{T/2} dx^0 e^{-ix^0(E_f - E_i)} . \tag{11.52}$$

Then, $(2\pi)\delta(0) \simeq T$, and the transition probability is infinite. Nevertheless, a finite number is obtained for the transition probability per time unit, which corresponds to the differential decay rate:

$$\frac{dP}{dx^0} = (2\pi)\delta(E_f - E_i)|\langle f|\mathcal{M}|i\rangle|^2 dQ = d\Gamma. \tag{11.53}$$

Unlike in the relativistic case, there are no additional factors needed to obtain $d\Gamma$. This is because we are already in the NR limit, where the states are normalized without any energy factors. See the analogous discussion in Sec. 5.5.1.

Eq. (11.53) yields our first example of a **Fermi's golden rule**. We now want to compute the total decay width associated with this radiative transition. Therefore, we sum over the final helicities of the photon and integrate over the whole phase space (note that the phase space integration measure for the photon is still the standard relativistic one), obtaining

$$\begin{aligned}\Gamma(n \to n'\gamma) &= \sum_\sigma \int d\tilde{k}(2\pi)\delta(E_f - E_i)|\langle f|\mathcal{M}|i\rangle|^2 \\ &= \frac{e^2\omega_k^3}{8\pi^2} \int d\Omega \sum_\sigma |\boldsymbol{\epsilon}_\sigma^* \cdot \mathbf{x}_{n'n}|^2 = \frac{e^2\omega_k^3}{3\pi}|\mathbf{x}_{n'n}|^2 ,\end{aligned} \tag{11.54}$$

where we have used the Ehrenfest theorem $i\langle n'|\mathbf{p}|n\rangle = m(E_n - E_{n'})\langle n'|\mathbf{x}|n\rangle \equiv m\omega_k \mathbf{x}_{n'n}$.

11.4.2 Interaction with a Classical Electromagnetic Field

Classical electromagnetic fields can, in some limits, be described using coherent states. Let us first recall some known results from quantum mechanics courses.

Exercise 1

Coherent states. Consider the following Hilbert space:

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0, \quad \left(a^\dagger\right)^n |0\rangle$$

a) Prove that the projector on the vacuum can be written in the following way

$$|0\rangle\langle 0| =: e^{-a^\dagger a} :$$

b) If we consider the states

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle$$

$$|c\rangle = \exp\left\{-\frac{1}{2}|c|^2\right\} \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}} |n\rangle$$

i) Determine $\langle n|n\rangle$ and $\langle c|c\rangle$.
ii) Prove that $a|c\rangle = c|c\rangle$.
iii) If we define $N = a^\dagger a$ compute $\bar{N} \equiv \langle c|N|c\rangle$ and

$$(\Delta N)^2 \equiv \langle c|N^2|c\rangle - (\langle c|N|c\rangle)^2$$

Using the results of this exercise, we are in the position to generate a classical electromagnetic field. To make this connection smoothly, it is convenient to work in a finite volume V (which quantizes the allowed values of the three-momentum $\mathbf{k}$ of the photon). For simplicity, we will consider V to be a square box (though our results will not depend on this), and work with the commutation relations:

$$[a_r(\mathbf{k}), a^\dagger_{r'}(\mathbf{k}')] = \delta_{rr'}\, \delta_{\mathbf{k}\mathbf{k}'} \,.$$

This means that the vector photon field reads

$$\mathbf{A} = \sum_{\mathbf{k},\, r} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_k}} \left(\boldsymbol{\epsilon}_r(\mathbf{k})\, a_r(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}} + \boldsymbol{\epsilon}^*_r(\mathbf{k})\, a^\dagger_r(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}\right) \,. \tag{11.55}$$

One then constructs the following coherent state:

$$|c\rangle \equiv |c(\mathbf{k}, r)\rangle = \exp\left\{-\frac{1}{2}|c|^2\right\} \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}} |n\rangle \,,$$

where

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger_r(\mathbf{k})\right)^n |0\rangle \,.$$

This coherent state has the same properties as the state $|c\rangle$ in the previous exercise, but generalized to an infinite number of degrees of freedom. Namely,

$$\langle c(\mathbf{k},r)|c(\mathbf{k}',r')\rangle = \delta_{r,r'}\delta_{\mathbf{k},\mathbf{k}'}\,, \qquad a_r^\dagger(\mathbf{k})|c(\mathbf{k}',r')\rangle = \delta_{r,r'}\delta_{\mathbf{k},\mathbf{k}'}c|c(\mathbf{k}',r')\rangle\,. \quad (11.56)$$

The average number of photons of this state is related to the absolute value of c:

$$\bar{N} = \langle c(\mathbf{k},r)|N|c(\mathbf{k}',r')\rangle = |c|^2\delta_{r,r'}\delta_{\mathbf{k},\mathbf{k}'} \quad (11.57)$$

and the root-mean-square fluctuation yields (for simplicity we take equal coherent states)

$$\Delta N = \sqrt{\langle c|N^2|c\rangle - \bar{N}^2} = |c| \quad (11.58)$$

This result is important. Even though $|c\rangle$ is not an eigenstate of N, the relative fluctuation in the number of photons reads

$$\frac{\Delta N}{\bar{N}} = \frac{1}{\sqrt{\bar{N}}}\,, \quad (11.59)$$

which goes to zero if $\bar{N} = |c|^2 \to \infty$.

Exercise 2

Compute $\langle c|\mathbf{A}|c\rangle$, $\langle c|\mathbf{E}|c\rangle$, and $(\Delta\mathbf{E})^2 \equiv \langle c|\mathbf{E}^2|c\rangle - (\langle c|\mathbf{E}|c\rangle)^2$.

The result of this exercise shows that the electromagnetic part of the matrix elements behaves as a classical field for large values of $|c|$. We illustrate this below with the following example.

Emission/Absorption of a photon by a hydrogen atom

We can now compute the total transition rate for absorption or emission of a photon by hydrogen-like systems in an external classical electromagnetic field characterized by a coherent state $|c(\mathbf{k},r)\rangle$. We interpret this as a hydrogen atom being in a medium full of photons. In some aspects, this computation is similar to the one performed in the previous section. On the other hand, it can be considered a kind of Compton scattering where the photon is classical (actually an external source).

We can take advantatge of the computation from the previous section. Basically, we only have to change the initial and final photon states. Therefore, we have

$$\begin{aligned}\langle f|S|i\rangle &= -\frac{ie}{m}\int_{-\infty}^{\infty} dx^0 \int d^d\mathbf{x}\langle c(\mathbf{k},r)|\mathbf{A}(x^0,\mathbf{x})|c(\mathbf{k},r)\rangle \\ &\quad \times\langle n'|\varphi^\dagger(x^0,\mathbf{x})(\boldsymbol{\nabla}\varphi(x^0,\mathbf{x}))|n\rangle\,, \end{aligned} \quad (11.60)$$

where the expression for $\mathbf{A}$ in the conventions of this section can be found in Eq. (11.55).

Using the properties of the photon coherent state and doing the integration over time, we obtain

$$\langle f|S|i\rangle = \frac{-ie}{m}\frac{1}{\sqrt{2\omega_k}}\left\{(2\pi)\delta(-\omega_k + E_n - E_{n'})\boldsymbol{\epsilon}_r(\mathbf{k})\frac{c_r^*}{\sqrt{V}}\int d^d\mathbf{x}\phi_{n'}^* e^{i\mathbf{k}\cdot\mathbf{x}}(\boldsymbol{\nabla}\phi_n(\mathbf{x}))\right.$$
$$\left. +(2\pi)\delta(\omega_k + E_n - E_{n'})\boldsymbol{\epsilon}_r^*(\mathbf{k})\frac{c_r}{\sqrt{V}}\int d^d\mathbf{x}\phi_{n'}^* e^{-i\mathbf{k}\cdot\mathbf{x}}(\boldsymbol{\nabla}\phi_n(\mathbf{x}))\right\}. \quad (11.61)$$

Note that, as in the previous section, we could use the multipole expansion for the matrix element if we wanted to. Nevertheless, we will keep the expressions general.

As in the previous section, there is no three-momentum conservation, since, at this level, there is no translational invariance. There are, however, important conceptual differences regarding the energy-conservation Dirac deltas. The reason is that we are working with coherent states, which do not have a well-defined energy. While the final energy has to be equal to the initial energy, neither is unique. From this result, we see that the coherent state acts as a reservoir of photons (or energy) that allows the hydrogen to change state. The classical electromagnetic field can act as a source or a sink of photons and two energy Dirac deltas appear, one for absorption, and the other for emission.

Following the same steps as in the previous section, we can compute the transition probability per time unit

$$\frac{dP}{dx^0} = \left(\frac{e}{m}\right)^2 \frac{1}{2\omega_k} \quad (11.62)$$
$$\times\left\{(2\pi)\,\delta(E_n - E_{n'} - \omega)\,\left|\langle n|\,e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}}\,\boldsymbol{\epsilon}(\mathbf{k})\cdot\hat{\mathbf{p}}\,|n'\rangle\right|^2\bar{n}\quad\text{(emission)}\right.$$
$$\left.+(2\pi)\,\delta(E_n - E_{n'} + \omega)\,\left|\langle n|\,e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}\,\boldsymbol{\epsilon}^*(\mathbf{k})\cdot\hat{\mathbf{p}}\,|n'\rangle\right|^2\bar{n}\quad\text{(absorption)}\right\},$$

where $\bar{n} = \frac{\bar{N}}{V}$ (recall that $|c|^2 = \langle c|\hat{N}|c\rangle = \bar{N}$). The first option is *emission* and the 2nd is *absorption* of the photon. One may ask how to obtain the infinite volume limit. One can take the limit $V \to \infty$ and $N \to \infty$ with $\bar{n} = \frac{N}{V} =$ *constant* and *finite*, i.e. with a finite photon number density per volume unit. This yields a finite limit for $dP/(dx^0)$, and another example of a **Fermi's Golden rule**.

Finally, we leave the reader with the following take-home exercise:

Exercise

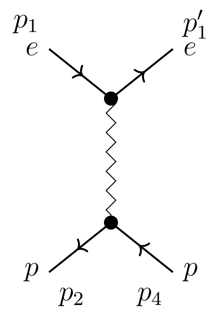
For the same coherent state, $|c(\mathbf{k},r)\rangle$ study the photoelectric effect (the ejection of an electron when the atom is placed in the radiation field).

11.5 Exercises

1. Consider the following correction to the Lagrangian density (11.12):

$$\delta\mathcal{L} = \varphi^\dagger(t,\mathbf{x})\frac{c_F^{(1)}}{2m_1}\boldsymbol{\sigma}\cdot e\mathbf{B}\varphi(t,\mathbf{x}) - \chi_c^\dagger(t,\mathbf{x})\frac{c_F^{(2)}}{2m_2}\boldsymbol{\sigma}\cdot e\mathbf{B}\chi_c(t,\mathbf{x})\,. \qquad (11.63)$$

Obtain the Feynman rules for this term and compute the following diagram



Obtain the associated correction to the potential and compute the hyperfine splitting for the case of hydrogen with $c_F^{(1)} = 1$ and $c_F^{(2)} = 1+\mu_p$, where μ_p is the anomalous magnetic moment of the proton. Compare the result with experiment.

2. Redo Sec. 11.4.1 with the Lagrangian

$$L = \int d^d\mathbf{x}\varphi^\dagger(t,\mathbf{x})\left(i\partial_0 + \frac{\boldsymbol{\nabla}^2}{2m} + \frac{\alpha}{|\mathbf{x}|} + e\mathbf{x}\cdot\mathbf{E}\right)\varphi(t,\mathbf{x}) - \int d^d\mathbf{x}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \qquad (11.64)$$

12 Quantum Gravity

The introduction of massless spin-two particles in a QFT interacting with other (massive or massless) particles parallels the introduction of massless spin-one particles we did in Chapters 7 and 9. There is nothing conceptually different from the point of view of the fundamentals of QFT.[1] As with the photon, a Lagrangian that includes graviton fields has a gauge symmetry. We need to consider a symmetric tensor $h_{\mu\nu}$ that, after some gauge fixing constraints, can be written in terms of a linear combination of the graviton creation and annihilation operators: $\hat{h}_{\mu\nu} \sim \hat{a} + \hat{a}^\dagger$, analogous to the photon one: $\hat{A}_\mu(x) \sim \hat{a} + \hat{a}^\dagger$. As in the photon case, besides explicit Lorentz invariant terms, non-local terms, such as those that appear in QED: $H_I \sim \int d^d\mathbf{x} \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$ show up to make the S-matrix Poincaré covariant. These terms combine with the physical graviton propagator to form a full propagator very much as in the QED case. In this book, we will not study quantum gravity. However, we cannot resist the temptation to give a small list of exercises that can already be solved with the tools presented in this book.

12.1 Exercises

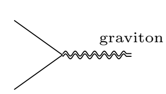
1. Using the generalized Feynman rules methods from Chapter 8 determine the propagator of the following Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\lambda h \partial^\lambda h \,,$$

   where $h_{\mu\nu}$ is a symmetric tensor that describes a graviton and $h \equiv h^\mu{}_\mu$.

2. Using the generalized Feynman rules methods from Chapter 8 determine the Feynman rule for the following vertex

$$\delta\mathcal{L} = c\sqrt{G} h_{\mu\nu} T^{\mu\nu} \qquad (12.1)$$



   where $T^{\mu\nu}$ is the energy-momentum tensor of a free scalar spin-zero particle, G is the Newton constant, and c is a dimensionless constant.

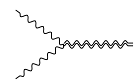
3. Using the results of the previous exercises, compute the scattering of two spinless and chargeless black holes with masses M_1 and M_2 at the lowest

[1]We do not address here issues such as the (perturbative) renormalizability of the theory.

non.vanishing order in $\sqrt{G}$, take the NR limit and fix the coefficient c such that the result agrees with Newton's law of gravitation. Discuss the similarities with Coulomb's law.

4. Using the generalized Feynman rules methods from Chapter 8 determine the Feynman rule for the following vertex

$$\delta\mathcal{L} = c'\sqrt{G}h_{\mu\nu}T^{\mu\nu} \qquad (12.2)$$



where $T^{\mu\nu}$ is the energy-momentum tensor of a free photon, and c' is a dimensionless constant.

Bibliography

[1] H. Goldstein, *Classical Mechanics*; Addison-Wesley Publishing Company, 2nd Edition (1980).

[2] A. Cornellà and J.I. Latorre, *Teoria clàssica de camps*; Universitat de Barcelona, 1984.

[3] D. Lurie, *Particles and Fields*; Interscience John Wiley & Sons. Inc. (1968)

[4] S. Weinberg, *The Quantum Theory of Fields*; Cambridge University Press (1995).

[5] L.H. Ryder, *Quantum Field Theory*; Cambridge University Press (1985).

[6] F.J. Yndurain, *Elements of group theory*; https://arxiv.org/pdf/0710.0468.

[7] C. Itzykson and J. Zuber, *Quantum Field Theory*; (International series in pure and applied physics) McGraw-Hill International Book Co (1987).

[8] B. Hatfield, *Quantum Field Theory of Point Particles and Strings*; Addison-Wesley Publishing Company,1989.

[9] S. Pokorsky, *Gauge Field Theories*; [Cambridge monographs on mathematical physics] (2000, Cambridge University Press), 2nd Edition.

[10] M. Peskin and D. Schroeder, *An introduction to Quantum Field Theory*; (Frontiers in Physics), theory-Addison-Wesley Pub. Co (1995).

[11] F. Mandl and G. Shaw, *Quantum Field Theory*; Interscience John Wiley & sons, Revised edition (1984).

[12] J.J. Sakurai, *Modern Quantum Mechanics*; Addison-Wesley Publishing Company, 1985.